TRIANGULAR TOEPLITZ CONTRACTIONS AND COWEN SETS FOR ANALYTIC POLYNOMIALS

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ABSTRACT. Let $\mathfrak{T}_N$ be the collection of $N \times N$ lower triangular Toeplitz contractions. We show that $\mathfrak{T}_N$ is compact and strictly convex in the spectral norm; that is, $\mathfrak{T}_N$ is compact, convex and

$$\partial \, \mathfrak{T}_N \subseteq \text{ext} \, \mathfrak{T}_N,$$

where $\partial(\cdot)$ and $\text{ext}(\cdot)$ denote the topological boundary and the set of extreme points, respectively. As an application, we show that the reduced Cowen set for an analytic polynomial is strictly convex; more precisely, if f is an analytic polynomial and if $G'_f := \{g \in H^\infty(\mathbb{T}) : g(0) = 0$ and the Toeplitz operator $T_{f+\bar{g}}$ is hyponormal$\}$, then G'_f is strictly convex. This answers a question of C. Cowen for the case of analytic polynomials.

1. INTRODUCTION

Let $M_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. If $A \in M_n(\mathbb{C})$ then the *singular values* of A are defined by the nonnegative square roots of the eigenvalues of A^*A. The *spectral norm* of A, $||A||$, is defined as the largest singular value of A, i.e., $||A|| = \max\{\lambda : \lambda \in \sigma(A^*A)\}$. It is well known that the spectral norm coincides with the operator norm, that is, $||A|| = \sup_{||x||=1} ||Ax||$. Thus, $||A|| \leq 1 \iff \sigma(A^*A) \subseteq [0,1]$, and $||A|| = 1 \iff 1 \in \sigma(A^*A)$. Let $\mathfrak{T}_N$ be the collection of $N \times N$ lower triangular *contractive Toeplitz matrices* (or simply *Toeplitz contractions*). An element A of $\mathfrak{T}_N$ is of the form

$$A = \begin{pmatrix} c_0 & 0 & 0 & \dots & & 0 \\ c_1 & c_0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & 0 \\ c_{N-2} & & & \ddots & \ddots & 0 \\ c_{N-1} & c_{N-2} & \dots & & c_1 & c_0 \end{pmatrix},$$

with $||A|| \leq 1$. In this article we establish that $\mathfrak{T}_N$ is strictly convex; that is, the topological boundary of $\mathfrak{T}_N$, $\partial \, \mathfrak{T}_N$, consists entirely of extreme points (Theorem 1). Our proof relies on the Carathéodory-Schur interpolation problem. As a consequence, we show that the reduced Cowen

set of an analytic polynomial is strictly convex (Theorem 9). This is related to the contractive completion problem for partially given triangular Toeplitz matrices, which has been considered by C.R. Johnson and L. Rodman [JR] and G. Nævdal [Næ].

2. Lower Triangular Toeplitz Contractions

Consider first the following interpolation problem, called the *Carathéodory–Schur interpolation problem* (CSIP). Given $c_0, c_1, \cdots, c_{N-1}$ in $\mathbb{C}$, find an analytic function k on the open unit disc $\mathbb{D}$ such that

   (i) $\widehat{k}(j) = c_j$, $\quad j = 0, \cdots, N-1$ $\quad$ ($\widehat{k}(j)$ denotes the j-th Fourier coefficient of k)

and

   (ii) $\sup_{z \in \mathbb{D}} |k(z)| \leq 1$.

CSIP can be analyzed by a matricial argument (cf. [Sch]): CSIP is solvable if and only if the Toeplitz matrix

$$C := \begin{pmatrix} c_0 & 0 & 0 & \ldots & 0 \\ c_1 & c_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{N-2} & & \ddots & \ddots & 0 \\ c_{N-1} & c_{N-2} & \ldots & c_1 & c_0 \end{pmatrix}$$

is a contraction, that is, $||C|| \leq 1$, or equivalently, $I - CC^* \geq 0$. Today this result is also called the Carathéodory–Fejér Theorem. A proof of this result can be obtained by means of the Commutant Lifting Theorem (cf. [GGK, Proposition XXVII.7.2]). In particular, by Pick's Theorem (cf. [Ga, Corollary 2.3]) CSIP has a unique solution k if and only if $\det(I - CC^*) = 0$. Moreover, in the cases where $\det(I - CC^*) = 0$, the unique solution is a finite Blaschke product k of the form

$$k(z) = e^{i\theta} \prod_{j=1}^{n} \frac{z - \zeta_j}{1 - \overline{\zeta_j} z} \quad (|\zeta_j| < 1 \text{ for } j = 1, \cdots, n; \ \theta \in [0, 2\pi)),$$

such that $\deg(k) = \operatorname{rank}(I - CC^*)$, where $\deg(k)$ denotes the *degree* of k, that is, the number of zeros of k in the open unit disc $\mathbb{D}$ (see also [Ta, Theorem]).

We now have:

Theorem 1. *Recall that $\mathfrak{T}_N$ is the collection of $N \times N$ lower triangular Toeplitz contractions. Then $\mathfrak{T}_N$ is compact and strictly convex in the spectral norm; that is, $\mathfrak{T}_N$ is compact, convex and*

$$\partial \mathfrak{T}_N \subseteq \operatorname{ext} \mathfrak{T}_N.$$

Proof. It is easy to see that $\mathfrak{T}_N$ is compact and convex. Moreover,

$$\partial \mathfrak{T}_N = \{C \in \mathfrak{T}_N : \ ||C|| = 1\}.$$

Towards strict convexity recall that $\mathcal{K}$ is a strictly convex subset of a vector space $\mathcal{X}$ if

$$x_1, x_2 \in \mathcal{X} \text{ and } \frac{1}{2}(x_1 + x_2) \in \partial \mathcal{K} \implies x_1 \notin \partial \mathcal{K}, \text{ or } x_2 \notin \partial \mathcal{K}, \text{ or } x_1 = x_2.$$

Equivalently, $\mathcal{K}$ is strictly convex if and only if $\partial \mathcal{K} \subseteq \operatorname{ext} \mathcal{K}$. Suppose now that $C, D, \frac{1}{2}(C+D) \in \partial \mathfrak{T}_N$. It will suffice to show that $C = D$. Write

$$
C := \begin{pmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_1 & c_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{N-2} & & \ddots & \ddots & 0 \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{pmatrix} \quad \text{and} \quad D := \begin{pmatrix} d_0 & 0 & 0 & \cdots & 0 \\ d_1 & d_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ d_{N-2} & & \ddots & \ddots & 0 \\ d_{N-1} & d_{N-2} & \cdots & d_1 & d_0 \end{pmatrix}
$$

Then by CSIP and the fact that $||C|| = ||D|| = 1$, there exist analytic functions B_i $(i = 1, 2, 3)$ on $\mathbb{D}$ such that

(1)
$$
\begin{cases} \widehat{B_1}(j) = c_j, \quad \widehat{B_2}(j) = d_j, \quad \widehat{B_3}(j) = \frac{1}{2}(c_j + d_j) \quad (j = 0, \cdots, N-1) \\ \sup_{z \in \mathbb{D}} |B_i(z)| \leq 1 \quad (i = 1, 2, 3). \end{cases}
$$

Now recall that if $A \in M_n$ with $||A|| = 1$ then $\det(I - AA^*) = 0$: indeed if $||A|| = 1$ then $||AA^*|| = 1$, so $1 \in \sigma(AA^*)$, forcing $\det(I - AA^*) = 0$. Thus by the preceding considerations and the assumption $||C|| = ||D|| = ||\frac{1}{2}(C+D)|| = 1$, we can see that each B_i is a finite Blaschke product, that is, each is the unique solution of the associated interpolation problem (1). But since $\sup_{z \in \mathbb{D}} |\frac{1}{2}(B_1 + B_2)(z)| \leq 1$ and $\frac{1}{2}\widehat{(B_1 + B_2)}(j) = \frac{1}{2}(c_j + d_j)$ $(j = 0, \cdots, N-1)$, it follows from uniqueness that $\frac{1}{2}(B_1 + B_2) = B_3$, which implies that $\frac{1}{2}(B_1 + B_2)$ is a finite Blaschke product. Write

$$
B_1 = e^{i\theta} \prod_{j=1}^{n} \frac{z - \zeta_j}{1 - \overline{\zeta_j} z} \quad \text{and} \quad B_2 = e^{i\omega} \prod_{j=1}^{m} \frac{z - \beta_j}{1 - \overline{\beta_j} z},
$$

where $|\zeta_j| < 1$ $(1 \leq j \leq n)$ and $|\beta_j| < 1$ $(1 \leq j \leq m)$. Observe that

$$
\frac{1}{2}(B_1 + B_2) = \frac{e^{i\theta} \prod_{j=1}^{n}(z - \zeta_j) \prod_{j=1}^{m}(1 - \overline{\beta_j} z) + e^{i\omega} \prod_{j=1}^{m}(z - \beta_j) \prod_{j=1}^{n}(1 - \overline{\zeta_j} z)}{2 \prod_{j=1}^{n}(1 - \overline{\zeta_j} z) \prod_{j=1}^{m}(1 - \overline{\beta_j} z)}.
$$

Since $\frac{1}{2}(B_1 + B_2)$ is a Blaschke product we have that for each ζ_j $(1 \leq j \leq n)$, either ζ_j is a zero of $\frac{1}{2}(B_1 + B_2)$ or $(1 - \overline{\zeta_j} z)$ is a factor of the numerator of $\frac{1}{2}(B_1 + B_2)$, and the same holds for each β_j $(1 \leq j \leq m)$. Thus this analysis shows that $m = n$ and $\{\zeta_j\}_{j=1}^{n} = \{\beta_j\}_{j=1}^{m}$. We can then write

$$
\frac{1}{2}(B_1 + B_2) = \frac{1}{2}(e^{i\theta} + e^{i\omega}) \prod_{j=1}^{n} \frac{z - \zeta_j}{1 - \overline{\zeta_j} z},
$$

which implies that $|\frac{1}{2}(e^{i\theta} + e^{i\omega})| = 1$, and hence $\theta = \omega$. Therefore we conclude that $B_1 = B_2$, so $c_j = d_j$ for $j = 0, \cdots, N-1$, which implies $C = D$. This completes the proof. $\square$

Although our proof of Theorem 1 relies on the the Carathéodory–Fejér Theorem, we expect that a (finite) matricial argument can be found.

Corollary 2. *For $S \in M_n(\mathbb{C})$, let $d_S := \text{dist}(S, \mathfrak{T}_N)$. Then*

(2) $$m_S := \{B \in \mathfrak{T}_N : \ ||S - B|| = d_S\}$$

is compact, convex and nonempty.

Proof. Since $m_0 = \{0\}$, we can assume $S \neq 0$. It is clear that m_S is closed and nonempty (by Theorem 1), so we shall focus on convexity. Let $B, C \in m_S$, and let $0 < t < 1$. Since $tB + (1 - t)C \in \mathfrak{T}_N$, we have

$$d_S \leq ||tB + (1 - t)C - S|| = ||t(B - S) + (1 - t)(C - S)||$$
$$\leq t\,||B - S|| + (1 - t)\,||C - S|| = d_S.$$

It follows that $tB + (1 - t)C \in m_S$, so m_S is convex. $\qquad\square$

Corollary 3. *Let A be a partial Toeplitz matrix whose strict upper triangular part is known and let tc_A denote the set of minimal Toeplitz completions $\widetilde{A}$ of A, i.e., (i) $\widetilde{A} - A$ is a Toeplitz contraction, and (ii) $||\widetilde{A}||$ has the smallest norm amongst Toeplitz completions. Then tc_A is either a singleton or an infinite set.*

Proof. Apply Corollary 2 with A in place of S. $\qquad\square$

Write $\mathfrak{N}$ for the collection of $N \times N$ upper triangular matrices. Note that $\mathfrak{N}$ forms a nest algebra with the nest $\{0\} \subseteq \{e_0\} \subseteq \cdots \subseteq \{e_0. \cdots, e_{N-1}\}$. Arveson's Distance Formula (cf. [Da, Theorem 9.5]) says that if $A = (c_{i-j}) \in \mathfrak{T}_N$ then

$$\text{dist}(A, \mathfrak{N}) = \sup_{0 \leq l \leq N-2} ||P_l^{\perp} A P_l|| = \max_{1 \leq l \leq N-1} ||(c_{l+i-j})_{1 \leq i \leq N-l, \ 1 \leq j \leq l}|| =: \alpha_c,$$

where P_l is the projection onto $\{e_0, \cdots, e_l\}$. But if $\mathfrak{N}$ is replaced by $\mathfrak{T}_N'$, the collection of $N \times N$ upper triangular *Toeplitz* contractions, then $\text{dist}(A, \mathfrak{T}_N') \geq \alpha_c$. Thus we need not expect that $\text{dist}(A, \mathfrak{T}_N') = \alpha_c$ if $N > 2$. However, if $A \in \mathfrak{T}_N$ satisfies $\text{dist}(A, \mathfrak{T}_N') = \alpha_c$ then by Corollary 3, the solution $S \in \mathfrak{T}_N'$ of the equation $\text{dist}(A, \mathfrak{T}_N') = ||A - S||$ is either unique or has infinitely many values. Thus we have:

Corollary 4. *Let A be a partial Toeplitz contraction whose strict upper triangular part is known, and assume that one of its rectangular submatrices has norm 1 (called the extremal case). Let $\widetilde{A}$ be a Toeplitz contraction of A such that $\widetilde{A} - A$ is a contraction itself. Then tc_A is either a singleton or an infinite set.*

Example 5. Let $T = \begin{pmatrix} b & a \\ x & b \end{pmatrix} \in M_2(\mathbb{C})$. By the preceding considerations there exists $x \in \mathbb{C}$ such that $\min ||T|| = || \begin{pmatrix} a \\ b \end{pmatrix} || = \sqrt{|a|^2 + |b|^2}$. Corollary 4 shows that x is unique or there are infinitely many solutions x. Indeed, if $b \neq 0$ then $x = -\frac{b^2}{|b|^2}\bar{a}$ is the only solution that minimizes $||T||$, and if $b = 0$ then all values x in the disk $|x| \leq |a|$ minimize $||T||$.

Remark 6. (i) We need not expect that the collection $\mathfrak{G}_N$ of $N \times N$ Toeplitz contractions is strictly convex: for example, take $C := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $D := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and observe that $C, D \in \partial \mathfrak{G}_N$ and also $\frac{1}{2}(C + D) \in \partial \mathfrak{G}_N$ because $||\frac{1}{2}(C + D)|| = || \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} || = 1$. Also the collection $\mathfrak{C}_N$ of $N \times N$ contractions is not strictly convex. In fact it is well known that $\mathrm{ext}\, \mathfrak{C}_N$ is the set of unitaries. More generally, it is known (cf. [Ha, Solution 136]) that the extreme points of the unit ball in the space of bounded linear operators on a Hilbert space are the maximal partial isometries, that is, the isometries and coisometries.

(ii) By (i) above,
$$\mathrm{ext}\, \mathfrak{T}_N \bigcap \mathrm{ext}\, \mathfrak{C}_N = \{e^{i\theta} I_N : \ \theta \in [0, 2\pi)\},$$
which also follows at once from the observation that unitary triangular Toeplitz matrices are rotations of the identity.

(iii) The *Frobenius norm* $|| \cdot ||_2$ is defined on M_n by

$$||A||_2 := \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

Then $[A, B] := \mathrm{tr}\,(AB^*)$ defines an inner product on $M_n(\mathbb{C})$, and the Frobenius norm is induced from $[\cdot, \cdot]$. Thus, the Frobenius norm satisfies the parallelogram identity

$$\frac{1}{2} \left(||C + D||_2^2 + ||C - D||_2^2 \right) = ||C||_2^2 + ||D||_2^2 \quad (C, D \in M_n(\mathbb{C})).$$

Therefore if $||C||_2 = ||D||_2 = ||\frac{1}{2}(C + D)||_2 = 1$ then it follows that $C = D$. This shows that $\mathfrak{T}_N$ is also strictly convex in the Frobenius norm.

3. COWEN SETS

In this section we will show how Theorem 1 can be applied to the study of Cowen's sets for analytic polynomials. We first need some definitions and preliminary results. A bounded linear operator A on a Hilbert space $\mathfrak{H}$ with inner product $(\cdot, \cdot)$ is said to be hyponormal if its selfcommutator $[A^*, A] = A^*A - AA^*$ induces a positive semidefinite quadratic form on $\mathfrak{H}$ via $\xi \mapsto ([A^*, A]\xi, \xi)$, for $\xi \in \mathfrak{H}$. When the Hilbert space under consideration is the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial \mathbb{D}$ in the complex plane, a natural collection of operators to study is the class of Toeplitz operators. Given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator T_φ with symbol φ is defined by $T_\varphi f := P(\varphi \cdot f)$, where $f \in H^2(\mathbb{T})$ and P denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. An elegant theorem of C. Cowen [Cow2] characterizes the hyponormality of a Toeplitz operator T_φ on $H^2(\mathbb{T})$ in terms of properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. We shall use a variant of Cowen's Theorem [Cow2], first proposed by Nakazi and Takahashi [NT].

Cowen's Theorem. *For $\varphi \in L^\infty(\mathbb{T})$, write*

$$\mathcal{E}(\varphi) := \left\{ k \in H^\infty(\mathbb{T}) : ||k||_\infty \leq 1 \ and \ \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T}) \right\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

Cowen's method is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution of a certain functional equation involving its symbol. Suppose φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, where $a_N \neq 0$. If $k(z) = \sum_{j=0}^{\infty} c_j z^j$ is a function in $H^\infty(\mathbb{T})$ then $\varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})$ if and only if the coefficients $c_0, \cdots, c_{N-1}$ are given by

$$(3) \qquad \begin{pmatrix} \overline{c_0} \\ \overline{c_1} \\ \vdots \\ \overline{c_{N-1}} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_{N-1} & a_N \\ a_2 & a_3 & \dots & a_N & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_N & 0 & \dots & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \overline{a_{-1}} \\ \overline{a_{-2}} \\ \vdots \\ \overline{a_{-N}} \end{pmatrix} \qquad \text{(cf. [Zhu]).}$$

By Cowen's Theorem, if $c_0, \cdots, c_{N-1}$ are given by (3) then the hyponormality of T_φ is equivalent to the existence of a function $k \in H^\infty(\mathbb{T})$ such that

$$(4) \qquad \begin{cases} \widehat{k}(j) = c_j, & j = 0, \cdots, N-1 \\ ||k||_\infty \leq 1, \end{cases}$$

which is precisely the formulation of CSIP.

From the preceding considerations the following result follows at once.

Lemma 7 ([HL, Theorem 1]). *Suppose that $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, where $a_N \neq 0$ and let $c_0, \cdots, c_{N-1}$ be given by (3). Then T_φ is hyponormal if and only if the Toeplitz matrix*

$$(5) \qquad C := \begin{pmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_1 & c_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{N-2} & & \ddots & \ddots & 0 \\ c_{N-1} & c_{N-2} & \dots & c_1 & c_0 \end{pmatrix}$$

is a contraction. Moreover, if $||C|| = 1$ then $\mathcal{E}(\varphi)$ has precisely one element k which is a finite Blaschke product of degree at most N.

We now introduce the notion of Cowen set.

Definition 8. (cf. [Cow1], [Cow2]) Let $H = \{h \in H^\infty : h(0) = 0 \text{ and } ||h||_2 \leq 1\}$. For $f \in H^2$, let G_f denote the set of g in H^2 such that for every h in H^2,

$$\sup_{h_0 \in H} |<hh_0, f>| \geq \sup_{h_0 \in H} |<hh_0, g>|.$$

We shall say that G_f is the *Cowen set* for f.

To see how this definition is relevant to hyponormality of Toeplitz operators, suppose $p \in H^\infty$ and $h \in H^2$. Then we have

$$\sup_{h_0 \in H} |<hh_0, p>| = \sup_{h_0 \in H} |<\bar{p}h, \bar{h}_0>| = \sup_{h_0 \in H} |<(I-P)\bar{p}h, \bar{h}_0>| = ||H_{\bar{p}}h||.$$

Recall ([Cow1, Proposition 11]) that if $f, g \in H^\infty$ and $\varphi = f + \bar{g}$ then the following are equivalent:

(1) T_φ is hyponormal;

(2) $||H_{\bar{f}}h|| \geq ||H_{\bar{g}}h||$ for every $h \in H^2$.

From Definition 8 and the above consideration we can see that if $f \in H^\infty$, then

$$G_f = \{g \in H^\infty : T_{f+\bar{g}} \text{ is hyponormal}\}.$$

In [Cow2], it was shown that G_f is balanced, convex, and weakly compact. Also, recall ([FL]) that if $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ is such that T_φ is hyponormal then $m \leq N$ and $|a_{-m}| \leq |a_N|$. Thus, if f is an analytic polynomial of degree N then

$$G_f = \{g : g \text{ is an analytic polynomial of degree at most } N \text{ and } T_{f+\bar{g}} \text{ is hyponormal}\}.$$

Since the hyponormality of T_φ is independent of the Fourier coefficient $\hat{\varphi}(0)$, we assume, without loss of generality, that $g(0) = 0$. Therefore, if $f(z) := \sum_{k=0}^{N} a_n z^N$, it is more convenient to work with the set

$$G'_f = \{g(z) := b_1 z + \cdots + b_N z^N : T_{f+\bar{g}} \text{ is hyponormal}\}.$$

We shall call G'_f the *reduced* Cowen set for f. We easily see that G'_f is convex and compact.

In 1988, Cowen posed the following:

Problem ([Cow1, Question 3]). *What are the extreme points of G'_f? In particular, if $g \in G'_f$ but $\lambda g \notin G'_f$ for all $|\lambda| > 1$, is g an extreme point of G'_f?*

We now answer the above problem for the case of analytic polynomials.

Theorem 9. *If f is an analytic polynomial of degree N then G'_f is strictly convex and*

$$\text{ext } G'_f = \{g(z) = b_1 z + \cdots + b_N z^N : ||C|| = 1\},$$

where C is the Toeplitz matrix in (5) corresponding to $\varphi = \bar{g} + f$.

Proof. Suppose $\varphi \equiv f + \bar{g}$ with $f(z) \equiv \sum_{j=0}^{N} a_j z^j$ and $g(z) \equiv \sum_{j=1}^{N} b_j z^j$, and write $G'_f := \{g : T_\varphi \text{ is hyponormal}\}$. By the preceding considerations, G'_f is convex and compact. We must show that $\partial G'_f \subseteq \text{ext } G'_f$. Suppose that $k(z) = \sum_{j=0}^{\infty} c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$, where $c_0, \cdots, c_{N-1}$ are given by (3). Write

$$\mathfrak{h}_N := \{(c_0, \cdots, c_{N-1}) \in \mathbb{C}^N : T_\varphi \text{ is hyponormal}\}.$$

If $F : b_1 z + \cdots + b_N z^N \mapsto (b_1, \cdots, b_N)$ is a canonical map from the set of polynomials g of degree at most N with $g(0) = 0$ onto $\mathbb{C}^N$ and if L is the inverse of the Hankel matrix in (3), then $LF(G'_f) = \overline{\mathfrak{h}_N}$, where $\overline{\mathfrak{h}_N}$ denotes the set of conjugates of elements in $\mathfrak{h}_N$. Note that LF is bijective. By [Con, Proposition V.7.9], every bijective bounded linear operator between Hilbert spaces preserves compactness, convexity, the topological boundary, and the set of extreme points, so it suffices to show that $\partial \mathfrak{h}_N \subseteq \text{ext } \mathfrak{h}_N$. By Lemma 7, we know that $\mathfrak{h}_N = \{(c_0, \cdots, c_{N-1}) \in \mathbb{C}^N : ||C|| \leq 1\}$, so that the result follows at once from Theorem 1. $\square$

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