Solution of the Truncated Hyperbolic Moment Problem

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Abstract. Let $Q(x, y) = 0$ be an hyperbola in the plane. Given real numbers $\beta \equiv \beta^{(2n)} = \{\beta_{ij}\}_{i,j \geq 0, i+j \leq 2n}$, with $\beta_{00} > 0$, the *truncated Q-hyperbolic moment problem* for β entails finding necessary and sufficient conditions for the existence of a positive Borel measure μ, supported in $Q(x, y) = 0$, such that $\beta_{ij} = \int y^i x^j \, d\mu$ $(0 \leq i + j \leq 2n)$. We prove that β admits a *Q-representing measure* μ (as above) if and only if the associated moment matrix $\mathcal{M}(n)(\beta)$ is positive semidefinite, recursively generated, has a column relation $Q(X, Y) = 0$, and the algebraic variety $\mathcal{V}(\beta)$ associated to β satisfies card $\mathcal{V}(\beta) \geq \text{rank } \mathcal{M}(n)(\beta)$. In this case, rank $\mathcal{M}(n) \leq 2n+1$; if rank $\mathcal{M}(n) \leq 2n$, then β admits a rank $\mathcal{M}(n)$-atomic (minimal) Q-representing measure; if rank $\mathcal{M}(n) = 2n + 1$, then β admits a Q-representing measure μ satisfying $2n + 1 \leq \text{card supp } \mu \leq 2n + 2$.

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1. Introduction

Let $Q(x, y) = 0$ denote an hyperbola in the plane. Given a real sequence $\beta \equiv \beta^{(2n)} : \beta_{00}, \beta_{01}, \beta_{10}, ..., \beta_{0,2n}, \beta_{2n,0}$, with $\beta_{00} > 0$, we seek concrete necessary and sufficient conditions so that there exists a positive Borel measure μ on $\mathbb{R}^2$ satisfying

$$\beta_{ij} = \int y^i x^j \, d\mu \qquad (i, j \geq 0, \ i + j \leq 2n) \qquad (1.1)$$

and

$$\operatorname{supp}\mu \subseteq \mathcal{Z}(Q) := \{(x,y) \in \mathbb{R}^2 : Q(x,y) = 0\}; \tag{1.2}$$

a measure μ satisfying (1.1) is a *representing measure* for β; μ is a *Q-representing measure* if it satisfies (1.1) and (1.2).

Our criterion for the existence of representing measures is expressed in terms of algebraic and geometric properties of the *moment matrix* $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ that we next describe. The size of $\mathcal{M}(n)$ is $m(n) := (n+1)(n+2)/2$, with rows and columns indexed as $1, X, Y, X^2, YX, Y^2, ..., X^n, YX^{n-1}, ..., Y^n$; the entry in row $Y^i X^j$, column $Y^k X^\ell$ of $\mathcal{M}(n)$ is $\beta_{i+k,j+\ell}$.

Let $\mathbb{R}_n[x,y]$ denote the space of real polynomials of degree at most n in two variables, and let $\mathcal{B}_n$ denote the basis consisting of the monomials in degree-lexicographic order, i.e., $\mathcal{B}_n : 1, x, y, x^2, yx, y^2, ..., x^n, yx^{n-1}, ..., y^n$. For $p \in \mathbb{R}_{2n}[x,y]$, $p(x,y) \equiv \sum a_{rs} y^r x^s$, let $\hat{p} := (a_{rs})$ denote the coefficient vector of p with respect to $\mathcal{B}_{2n}$. Further, let $L_\beta : \mathbb{R}_{2n}[x,y] \to \mathbb{R}$ be the *Riesz functional* defined by $L_\beta(p) := \sum a_{rs} \beta_{rs}$; then $\mathcal{M}(n)$ is uniquely determined by

$$(\mathcal{M}(n)\hat{p}, \hat{q}) := L_\beta(pq) \ \ (p, q \in \mathbb{R}_n[x,y]). \tag{1.3}$$

In particular, if μ is a representing measure for β, then $(\mathcal{M}(n)\hat{p}, \hat{p}) = L_\beta(p^2) = \int p^2 \, d\mu \geq 0$. Since $\mathcal{M}(n)$ is real symmetric, it follows that $(\mathcal{M}(n)(\hat{p} + i\hat{q}), \hat{p} + i\hat{q}) \geq 0$, whence

$$\mathcal{M}(n) \geq 0 \tag{1.4}$$

(i.e., $\mathcal{M}(n)$ is a positive semi-definite operator on $\mathbb{C}^{m(n)}$).

For $p \in \mathbb{R}_n[x,y]$, $p(x,y) \equiv \sum a_{ij} y^i x^j$, we define an element $p(X,Y)$ of $\mathcal{C}_{\mathcal{M}(n)}$, the column space of $\mathcal{M}(n)$, by $p(X,Y) := \sum a_{ij} Y^i X^j$; for polynomials p and q with $\deg p + \deg q \leq n$, we also write $p(X,Y)q(X,Y)$ for $(pq)(X,Y)$. It follows from [CuFi1, Proposition 3.1] that if μ is a representing measure for β, then

$$\text{for } p \in \mathbb{R}_n[x,y], \ p(X,Y) = 0 \Leftrightarrow \operatorname{supp}\mu \subseteq \mathcal{Z}(p). \tag{1.5}$$

It follows immediately from (1.5) that if β has a representing measure, then $\mathcal{M}(n)$ is *recursively generated* in the following sense:

$$p, q, pq \in \mathbb{R}_n[x,y], \ p(X,Y) = 0 \Longrightarrow (pq)(X,Y) = 0. \tag{1.6}$$

We define the *variety* of $\mathcal{M}(n)$ (or of β) by $\mathcal{V}(\mathcal{M}(n)) := \bigcap_{\substack{p \in \mathbb{R}_n[x,y] \\ p(X,Y)=0}} \mathcal{Z}(p)$; [CuFi1, Proposition 3.1 and Corollary 3.7] implies that if μ is a representing measure for β, then $\operatorname{supp}\mu \subseteq \mathcal{V}(\mathcal{M}(n))$ and $\operatorname{rank}\mathcal{M}(n) \leq \operatorname{card}\operatorname{supp}\mu \leq \operatorname{card}\mathcal{V}(\mathcal{M}(n))$, whence $\mathcal{M}(n)$ satisfies the *variety condition*

$$\operatorname{rank}\mathcal{M}(n) \leq \operatorname{card}\mathcal{V}(\mathcal{M}(n)). \tag{1.7}$$

In the sequel we repeatedly reply on the following basic result of [CuFi1, Theorem 5.13]:

$$\beta \text{ admits a } \operatorname{rank} \mathcal{M}(n)\text{-atomic (minimal) representing measure} \tag{1.8}$$

$$\text{if and only if } \mathcal{M}(n) \geq 0 \text{ and } \mathcal{M}(n) \text{ admits an extension to}$$

$$\text{a (necessarily positive) moment matrix } \mathcal{M}(n+1) \text{ satisfying}$$

$$\operatorname{rank} \mathcal{M}(n+1) = \operatorname{rank} \mathcal{M}(n);$$

we refer to such an extension as a *flat extension*.

   Our main result shows that properties (1.4) - (1.7) completely characterize the existence of Q-representing measures, as follows.

Theorem 1.1. *Let* $Q(x,y) = 0$ *be an hyperbola in the plane. A sequence* $\beta \equiv \beta^{(2n)}$ *has a representing measure supported in* $Q(x,y) = 0$ *if and only if* $\mathcal{M}(n)$ *is positive semi-definite, recursively generated,* $Q(X,Y) = 0$ *in* $\mathcal{C}_{\mathcal{M}(n)}$, *and* $\operatorname{rank} \mathcal{M}(n) \leq \operatorname{card} \mathcal{V}(\mathcal{M}(n))$. *In this case,* $\operatorname{rank} \mathcal{M}(n) \leq 2n+1$; *if* $\operatorname{rank} \mathcal{M}(n) \leq 2n$, *then there is a* $\operatorname{rank} \mathcal{M}(n)$-*atomic* Q-*representing measure, while if* $\operatorname{rank} \mathcal{M}(n) = 2n+1$, *there is a* Q-*representing measure* μ *for which* $2n+1 \leq \operatorname{card} \operatorname{supp} \mu \leq 2n+2$.

   Consider the following property for a polynomial $P \in \mathbb{R}_n[x,y]$:

$$\beta \equiv \beta^{(2n)} \text{ has a representing measure supported in } \mathcal{Z}(P) \text{ if and only if} \tag{A'_n}$$

$$\mathcal{M}(n)(\beta) \text{ is positive semi-definite, recursively generated,}$$

$$P(X,Y) = 0 \text{ in } \mathcal{C}_{\mathcal{M}(n)}, \text{ and } \operatorname{rank} \mathcal{M}(n) \leq \operatorname{card} \mathcal{V}(\mathcal{M}(n)).$$

Polynomials which satisfy (A'_n) form an attractive class, because if P satisfies (A'_n), then the degree-$2n$ moment problem on $P(x,y) = 0$ can be solved by concrete tests involving only elementary linear algebra and the calculation of roots of polynomials. Theorem 1.1 shows that each hyperbolic polynomial satisfies (A'_n) for $n \geq 2$. Moreover, P satisfies (A'_n) for $n \geq \deg P$ if P represents a line [CuFi2], ellipse [CuFi5], or parabola [CuFi7]. These results together yield the following.

Theorem 1.2. *If* $\deg P \leq 2$, *then* P *satisfies* (A'_n) *for every* $n \geq \deg P$.

Despite Theorem 1.2, there are differences between the parabolic and elliptic moment problems and the hyperbolic problem. In the former cases, the conditions of (A'_n) always imply the existence of a $\operatorname{rank} \mathcal{M}(n)$-atomic representing measure, corresponding to a flat extension of $\mathcal{M}(n)$; for this reason, positive Borel measures supported on these curves always admit *Gaussian* cubature rules, i.e., $\operatorname{rank} \mathcal{M}(n)$-atomic cubature rules of degree $2n$ (cf. [FiPe]). By contrast, in the hyperbolic case, minimal representing measures μ sometimes entail $\operatorname{card} \operatorname{supp} \mu > \operatorname{rank} \mathcal{M}(n)$ (and Gaussian cubature rules may fail to exist; cf. Example 5.4).

   The preceding results are part of a general study of truncated moment problems that we initiated in [CuFi1], and are also related to the classical full moment problem, where moments of all orders are prescribed, i.e., $\beta \equiv \beta^{(\infty)} = (\beta_{ij})_{i,j \geq 0}$ (cf. [AhKr], [Akh], [KrNu], [PuVa], [Schm1], [ShTa], [StSz2]). Theorem 1.2 is motivated in part by results of J. Stochel [Sto1], who solved the full moment problem

on planar curves of degree at most 2. Paraphrasing [Sto1] (i.e., translating from the language of *moment sequences* into the language of moment matrices), we consider the following property of a polynomial P:

$$\beta^{(\infty)} \text{ has a representing measure supported in } P(x,y) = 0 \qquad (A)$$
$$\text{if and only if } \mathcal{M}(\infty)(\beta) \geq 0 \text{ and } P(X,Y) = 0 \text{ in } \mathcal{C}_{\mathcal{M}(\infty)}.$$

Theorem 1.3. *(Stochel* [Sto1]*) If* $\deg P \leq 2$, *then* P *satisfies* (A).

In [Sto1], Stochel also proved that there exist polynomials of degree 3 that do not satisfy (A); Stochel and F. Szafraniec [StSz1] proved that there are polynomials of arbitrarily large degree that satisfy (A) (cf. [Fia3]). Whether there exists a polynomial P such that P fails to satisfy (A'_n) for some $n \geq \deg P$ is an open problem (cf. Section 6).

The link between the truncated and full moment problems is provided by another result of Stochel (which actually holds for moment problems on $\mathbb{R}^d$, $d > 1$).

Theorem 1.4. *(cf.* [Sto2]*)* $\beta^{(\infty)}$ *has a representing measure supported in a closed set* $K \subseteq \mathbb{R}^2$ *if and only if, for each* n, $\beta^{(2n)}$ *has a representing measure supported in* K.

In Section 6 we will use Theorem 1.4 to give a new proof of Theorem 1.3. To do so, we require the following refinement of Theorem 1.1, which relates the existence of representing measures to extensions of moment matrices.

Theorem 1.5. *Let* $H := \{(x,y) \in \mathbb{R}^2 : Q(x,y) = 0\}$ *be an hyperbola. For* $\beta \equiv \beta^{(2n)}$, *assume that* $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ *is positive, recursively generated, and satisfies* $Q(X,Y) = 0$ *in* $\mathcal{C}_{\mathcal{M}(n)}$. *Then* $\operatorname{rank} \mathcal{M}(n) \leq 2n + 1$, *and the following statements are equivalent.*

(i) β *admits a representing measure (necessarily supported in* H).
(ii) β *admits a representing measure (necessarily supported in* H) *with convergent moments of degree up to* $2n + 2$.
(iii) β *admits a representing measure* μ *(necessarily supported in* H) *satisfying* $\operatorname{card supp} \mu \leq 1 + \operatorname{rank} M(n)$. *If* $\operatorname{rank} M(n) \leq 2n$, *then* μ *can be taken so that* $\operatorname{card supp} \mu = \operatorname{rank} M(n)$.
(iv) $M(n)$ *admits a positive, recursively generated moment matrix extension* $M(n+1)$.
(v) $M(n)$ *admits a positive, recursively generated extension* $M(n+1)$, *with* $\operatorname{rank} M(n+1) \leq 1 + \operatorname{rank} M(n)$, *and* $M(n+1)$ *admits a flat extension* $M(n+2)$. *If* $\operatorname{rank} M(n) \leq 2n$, *then* $M(n)$ *admits a flat extension* $M(n+1)$.
(vi) $\operatorname{rank} M(n) \leq \operatorname{card} V(M(n))$.

Condition (vi) in Theorem 1.5 is the concrete condition which, together with positivity and recursiveness, provides an effective test for the existence of representing measures; we illustrate Theorem 1.5 with an example.

Example 1.6. We consider a case of $M(2)$ satisfying $YX = 1$; let

$$\mathcal{M}(2)(\beta) := \begin{pmatrix} 1 & 0 & 0 & a & 1 & a \\ 0 & a & 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 0 \\ a & 0 & 0 & c & a & 1 \\ 1 & 0 & 0 & a & 1 & a \\ a & 0 & 0 & 1 & a & c \end{pmatrix}.$$

By calculating nested determinants, we see at once that the 3 by 3 upper left-hand corner is positive and invertible if and only if $a > 1$, and that the 4 by 4 upper left-hand corner is positive and invertible if and only if $c > a^2$. We now let $a > 1$ and $c := a^2 + r$, where $r > 0$. Then rank $M(2) = 4$ if and only if

$$(a^2 - 1)(r^2 - a^2 + 1) = 0. \tag{1.9}$$

 The positive root of (1.9) is $r_1 := \sqrt{a^2 - 1}$, so we set $r := r_1$ and observe that in $C_{\mathcal{M}(2)}$, $Y^2 = 2a1 - X^2$. Since $YX = 1$, we next find the variety $V(\beta)$ by solving the pair of equations

$$\begin{cases} y^2 = 2a - x^2 \\ \quad y = \frac{1}{x} \end{cases}. \tag{1.10}$$

It is easy to see that (1.10) has exactly four roots, $\{(x_i, y_i)\}_{i=1}^4$, where $x_1 := -(a - \sqrt{a^2 - 1})$, $x_2 := a - \sqrt{a^2 - 1}$, $x_3 := -(a + \sqrt{a^2 - 1})$, $x_4 := a + \sqrt{a^2 - 1}$, and $y_i := \frac{1}{x_i}$ $(i = 1, 2, 3, 4)$. Thus, rank $M(2) = 4 = \text{card } V(M(2))$. According to Theorem 1.5(vi)$\Rightarrow$(i), $\beta^{(4)}$ admits a representing measure μ. Since $4 = \text{rank } M(2) \leq \text{card supp } \mu \leq \text{card } V(\beta) = 4$, it follows that $\text{supp } \mu = V(\beta)$ and that $M(2)$ admits a flat extension $M(3)$ (cf. (1.8)). $\qquad\square$

Theorem 1.5 shows that minimal Q-representing measures for $\beta^{(2n)}$ arise either from flat extensions of $\mathcal{M}(n)$ or of $\mathcal{M}(n+1)$ (cf. [Fia2]). In the presence of a flat extension, there is a simple procedure for computing the atoms and densities of a corresponding minimal representing measure.

Theorem 1.7. *(cf.* [CuFi8, Theorem 2.21]*) If $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ is positive semidefinite and admits a flat extension $\mathcal{M}(n + 1)$, then $\mathcal{V} := \mathcal{V}(\mathcal{M}(n + 1))$ satisfies card $\mathcal{V} = r$ $(\equiv \text{rank } \mathcal{M}(n))$, and $\mathcal{V} \equiv \{(x_i, y_i)\}_{i=1}^r \subseteq \mathbb{R}^2$ forms the support of the unique representing measure μ for $\mathcal{M}(n + 1)$, i.e., μ is of the form $\mu = \sum_{i=1}^r \rho_i \delta_{(x_i, y_i)}$ with $\rho_i > 0$ $(1 \leq i \leq r)$. If $\mathcal{B} \equiv \{Y^{i_k} X^{j_k}\}_{k=1}^r$ is a maximal linearly independent subset of columns of $\mathcal{M}(n)$, let V be the $r \times r$ matrix whose entry in row k, column ℓ is $y_\ell^{i_k} x_\ell^{j_k}$ $(1 \leq k, \ell \leq r)$. Then V is invertible, and $\rho \equiv (\rho_1, ..., \rho_r)$ is uniquely determined by $V\rho^t = (\beta_{i_1, j_1}, ..., \beta_{i_r, j_r})^t$.*

Example 1.8. (Example 1.6 cont.) We now use Theorem 1.7 to compute the densities for the measure $\mu := \sum_{i=1}^4 \rho_i \delta_{(x_i, y_i)}$ of Example 1.6. Since $\{1, X, Y, X^2\}$

is a basis for $\mathcal{C}_{\mathcal{M}(2)}$, Theorem 1.7 implies that

$$V \equiv \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{pmatrix},$$

is invertible (indeed, $\det V = (x_1^2 - x_3^2)^2)$. The vector $\mathbf{v} := (\beta_{i_1,j_1}, ..., \beta_{i_r,j_r})^t$ in Theorem 1.7 is $(1, 0, 0, a)^t$, so a calculation of $\rho = V^{-1}\mathbf{v}$ shows that $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \frac{1}{4}$. $\qquad\square$

To prove Theorems 1.1, 1.2 and 1.5, we will reduce the analysis of truncated moment problems on conics to the study of truncated moment problems on four special conics: $x^2 + y^2 = 1$, $y = x^2$, $yx = 1$, and $yx = 0$. This reduction was initially described (using complex moment matrices) in [CuFi6], but we now require a more detailed analysis.

For $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, with $b_1c_2 \neq b_2c_1$, let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ denote the degree-one bijection defined by $\Phi(x, y) \equiv (\varphi_1, \varphi_2) := (a_1 + b_1x + c_1y, a_2 + b_2x + c_2y)$ $(x, y \in \mathbb{R})$. Given $\beta^{(2n)}$, define $\tilde{\beta}^{(2n)}$ by $\tilde{\beta}_{ij} := L_\beta(\varphi_2^i \varphi_1^j)$ $(0 \le i + j \le 2n)$, where L_β denotes the Riesz functional associated with β. It is straightforward to verify that $L_{\tilde{\beta}}(p) = L_\beta(p \circ \Phi)$ for every $p \in \mathbb{R}_n[x, y]$. (Note that for $p(x, y) \equiv \sum a_{ij}y^i x^j$, $(p \circ \Phi)(x, y) = p(\varphi_1, \varphi_2) \equiv \sum a_{ij}\varphi_2^i \varphi_1^j$.) Also, $L_\beta(p) = L_{\tilde{\beta}}(p \circ \Psi)$, where $\Psi := \Phi^{-1}$, given by $\Psi(u, v) := \frac{1}{b_1c_2 - b_2c_1}(c_2(u - a_1) - c_1(v - a_2), -b_2(u - a_1) + b_1(v - a_2))$.

Let $\mathcal{H}^{(n)} := \{\hat{p} : p \in \mathbb{R}_n[x, y]\} \cong \mathbb{R}^{m(n)}$ and consider the linear map $J \equiv J^{(n)} : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$ defined by $J(\hat{p}) := \widehat{p \circ \Phi}$ $(p \in \mathbb{R}_n[x, y])$. The map J is invertible, with inverse $J^{-1}(\hat{q}) := \widehat{q \circ \Psi}$ $(q \in \mathbb{R}_n[x, y])$. Note that $\mathcal{H}^{(n)}$ admits a vector space direct sum decomposition $\mathcal{H}^{(n)} = \mathcal{H}^{(n-1)} \bigoplus \mathcal{H}_n$, where $\mathcal{H}_n$ is the subspace spanned by vectors $\widehat{y^i x^j}$ with $i + j = n$. Since $J^{(n)}$ and $J^{(n-1)} : \mathcal{H}^{(n-1)} \to \mathcal{H}^{(n-1)}$ are both invertible, relative to the above decomposition $J^{(n)}$ can be represented as

$$J^{(n)} = \begin{pmatrix} J^{(n-1)} & K_n \\ 0 & L_n \end{pmatrix}, \qquad (1.11)$$

and thus $(J^{(n)})^{-1}$ admits a similar triangular representation.

Proposition 1.9. (Invariance under degree-one transformations.) *Let $\mathcal{M}(n)$ and $\tilde{\mathcal{M}}(n)$ be the moment matrices associated with β and $\tilde{\beta}$.*

(i) *$\tilde{\mathcal{M}}(n) = J^*\mathcal{M}(n)J$.*

(ii) *J is invertible.*

(iii) *$\tilde{\mathcal{M}}(n) \ge 0 \Leftrightarrow \mathcal{M}(n) \ge 0$.*

(iv) *$\operatorname{rank} \tilde{\mathcal{M}}(n) = \operatorname{rank} \mathcal{M}(n)$.*

(v) *The formulas $\mu = \tilde{\mu} \circ \Phi$ and $\tilde{\mu} = \mu \circ \Psi$ establish a one-to-one correspondence between the sets of representing measures for β and $\tilde{\beta}$, which preserves measure class and cardinality of the support; moreover, $\Phi(\operatorname{supp} \mu) = \operatorname{supp} \tilde{\mu}$ and $\Psi(\operatorname{supp} \tilde{\mu}) = \operatorname{supp} \mu$.*

(vi) *For $p \in \mathbb{R}_n[x, y]$, $p\left(\tilde{X}, \tilde{Y}\right) = J^*\left((p \circ \Phi)(X, Y)\right)$ and $p(X, Y) =$*
   $(J^{-1})^*((p \circ \Psi)(\tilde{X}, \tilde{Y}))$.

(vii) *$\tilde{\mathcal{M}}(n)$ is recursively generated if and only if $\mathcal{M}(n)$ is recursively generated.*

(viii) *$\Phi(\mathcal{V}(\beta)) = \mathcal{V}(\tilde{\beta})$ and $\mathcal{V}(\beta) = \Psi(\mathcal{V}(\tilde{\beta}))$.*

(ix) *$\mathcal{M}(n)$ is positive semi-definite and admits a positive, recursively generated (resp. flat) extension $\mathcal{M}(n+1)$ if and only if $\tilde{\mathcal{M}}(n)$ is positive semi-definite and admits a positive, recursively generated (resp. flat) extension $\tilde{\mathcal{M}}(n+1)$.*

Proof. We omit the proofs of (i) - (viii), which are straightforward. For (ix), suppose $\mathcal{M}(n)$ is positive and admits a positive, recursively generated extension

$$\mathcal{M}(n+1)(\check{\beta}) \equiv \begin{pmatrix} \mathcal{M}(n) & B(n+1) \\ B(n+1)^t & C(n+1) \end{pmatrix}.$$

Part (i) (using $n+1$ instead of n) and (1.11) imply that

$$\mathcal{M} \equiv \tilde{\mathcal{M}}(n+1) = (J^{(n+1)})^* \mathcal{M}(n+1)(\check{\beta}) J^{(n+1)}$$
$$= \begin{pmatrix} (J^{(n)})^* \mathcal{M}(n) J^{(n)} & * \\ * & * \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{M}}(n) & * \\ * & * \end{pmatrix}.$$

Thus, $\mathcal{M}$ is a positive moment matrix extension of $\tilde{\mathcal{M}}(n)$, and (vii) (applied with $n+1$) implies that $\mathcal{M}$ is recursively generated. In the case when $\mathcal{M}$ is a flat extension, we have, from (i), (ii) and (iv), $\operatorname{rank}\tilde{\mathcal{M}}(n) = \operatorname{rank}\mathcal{M}(n) = \operatorname{rank}\mathcal{M}(n+1)(\check{\beta}) = \operatorname{rank}\mathcal{M}$, so $\mathcal{M}$ is a flat extension of $\tilde{\mathcal{M}}(n)$. The converse is proved in the same way. $\square$

It is well known that a general conic $Q(x, y) = 0$ can be transformed into one of the four special cases by means of rotation, scaling and translation; thus, there is a degree-one map Φ (as above) such that $Q \circ \Phi$ is a special conic. Note from Proposition 1.9 that such properties of $\mathcal{M}(n)$ as positivity, recursiveness, the variety condition, and the existence of flat or positive extensions $\mathcal{M}(n+1)$ are invariant under degree-one mappings, which also preserve the existence of representing measures and the cardinality of the support. These observations show that Theorems 1.1 and 1.5 are valid for arbitrary hyperbolas if and only if they are valid for $yx = 1$ and $yx = 0$. We prove Theorem 1.1 for $yx = 1$ in Section 2 (Theorem 2.1) and for $yx = 0$ in Section 3 (Theorem 3.1). We prove Theorem 1.5 (for both hyperbolas) in Section 4 (Theorem 4.1). Section 5 contains examples illustrating our results. Section 6 contains a new proof of Theorem 1.3, based on moment matrix techniques, including Theorem 1.5.

The remainder of this section is devoted to notation and basic results concerning real moment matrices. Given a collection $\beta^{(2n)} : \beta_{00}, \beta_{01}, \beta_{10}, ..., \beta_{0,2n}, ..., \beta_{2n,0}$, we can describe $\mathcal{M}(n)(\beta)$ by means of a block matrix decomposition $\mathcal{M}(n)(\beta) :=$

$(\mathcal{M}_{ij}(\beta))_{i,j=0}^n$, where

$$\mathcal{M}_{ij}(\beta) := \begin{pmatrix} \beta_{0,i+j} & \beta_{1,i+j-1} & \cdots & \beta_{j,i} \\ \beta_{1,i+j-1} & \beta_{2,i+j-2} & \cdots & \beta_{j+1,i-1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{i,j} & \beta_{i+1,j-1} & \cdots & \beta_{i+j,0} \end{pmatrix}.$$

Recall that $\mathcal{M}(n)$ has size $m(n) \equiv \frac{(n+1)(n+2)}{2}$. For any matrix M of this size, $[M]_k$ denotes the compression of M to the first k rows and columns; similarly, for a vector $\mathbf{v}$, $[\mathbf{v}]_k$ denotes the compression of $\mathbf{v}$ to the first k entries. We also consider compressions of M and $\mathbf{v}$ to a set E of rows and columns, and denote such compressions by $[M]_E$ and $[\mathbf{v}]_E$. For $i+j, k+\ell \leq n$, $\langle Y^i X^j, Y^k X^l \rangle_M$ (or simply $\langle Y^i X^j, Y^k X^l \rangle$) denotes the entry of M in row $Y^k X^l$ and column $Y^i X^j$.

A theorem of Smul'jan [Smu] shows that a block matrix

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \tag{1.12}$$

is positive if and only if (i) $A \geq 0$, (ii) there exists a matrix W such that $B = AW$, and (iii) $C \geq W^* AW$ (since $A = A^*$, $W^* AW$ is independent of W satisfying $B = AW$). Note also that if $M \geq 0$, then rank $M = $ rank A if and only if $C = W^* AW$; conversely, if $A \geq 0$ and there exists W such that $B = AW$ and $C = W^* AW$, then $M \geq 0$ and rank $M = $ rank A. In the sequel, for $A \geq 0$, we refer to M as an *extension* of A, and as a *flat extension* if rank $M = $ rank A. Thus, a flat extension of a positive matrix A is completely determined by a choice of block B satisfying $B = AW$ and $C = W^* AW$ for some matrix W; we denote such an extension by $[A; B]$. It follows from the Extension Principle [Fia1] that if $M \geq 0$, then column dependence relations in A extend to $\begin{pmatrix} A \\ B^* \end{pmatrix}$; further, if M is a flat extension of A (≥ 0), then column dependence relations in $(A \ B)$ extend to M.

For an $(n+1) \times (n+2)$ moment matrix block $B_{n,n+1}$, representing "new moments" of degree $2n+1$ for a prospective representing measure of $\beta^{(2n)}$, let

$$B(n+1) := \begin{pmatrix} B_{0,n+1} \\ \vdots \\ B_{n-1,n+1} \\ B_{n,n+1} \end{pmatrix}.$$

By Smul'jan's theorem, $\mathcal{M}(n) \geq 0$ admits a (necessarily positive) flat extension

$$[\mathcal{M}(n); B] = \begin{pmatrix} \mathcal{M}(n) & B \\ B^* & C \end{pmatrix}$$

in the form of a moment matrix $\mathcal{M}(n+1)$ if and only if

$$B = B(n+1) \text{ and } B = \mathcal{M}(n)W \text{ for some } W \tag{1.13}$$

(i.e., Ran $B \subseteq$ Ran $\mathcal{M}(n)$[Dou]); and

$C := W^* \mathcal{M}(n)W$ is Hankel

(i.e., C has the form of a moment matrix block $B_{n+1,n+1}$).

2. The Truncated Moment Problem on Nondegenerate Hyperbolas

In this section we prove Theorem 1.1 for nondegenerate hyperbolas. In view of Proposition 1.9 and the remarks following it, it suffices to consider the case $yx = 1$. The necessity of the conditions in Theorem 1.1 is clear from Section 1, and sufficiency follows immediately from Theorem 2.1 below.

Theorem 2.1. *Let $\beta \equiv \beta^{(2n)} : \beta_{00}, \beta_{01}, \beta_{10}, ..., \beta_{0,2n}, ..., \beta_{2n,0}$ be a family of real numbers, $\beta_{00} > 0$, and let $\mathcal{M}(n)$ be the associated moment matrix. Assume that $\mathcal{M}(n)$ is positive, recursively generated, and satisfies $YX = 1$ and rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\beta)$. Then rank $\mathcal{M}(n) \leq 2n+1$. If rank $\mathcal{M}(n) \leq 2n$, then $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$ (so β admits a rank $\mathcal{M}(n)$-atomic representing measure supported in $yx = 1$). If rank $\mathcal{M}(n) = 2n+1$, then $\mathcal{M}(n)$ admits an extension to a positive, recursively generated extension $\mathcal{M}(n+1)$, satisfying $2n+1 \leq$ rank $\mathcal{M}(n+1) \leq 2n+2$, and $\mathcal{M}(n+1)$ admits a flat extension $\mathcal{M}(n+2)$ (so β admits a representing measure μ supported in $yx = 1$, with $2n+1 \leq$ card supp $\mu \leq 2n+2$.*

We require several preliminary results for the proof of Theorem 2.1. By [CuFi2, Theorem 2.1], we know that $\mathcal{M}(n)$ (positive semi-definite and recursively generated) admits flat extensions when $\{1, X, Y\}$ is linearly dependent in $\mathcal{C}_{\mathcal{M}(n)}$. Thus, hereafter we will assume that $\{1, X, Y\}$ is linearly independent. We begin with an elementary lemma that exploits the fact that $\mathcal{M}(n)$ is recursively generated. For $1 \leq k \leq n$ let

$$\mathcal{S}_n(k) := \{1, X, Y, X^2, Y^2, ..., X^k, Y^k\} \subseteq \mathcal{C}_{\mathcal{M}(n)}.$$

Lemma 2.2. *For $n \geq 2$, let $\mathcal{M}(n)$ be positive and recursively generated, and assume that $YX = 1$. Then each column of $\mathcal{M}(n)$ is equal to a column in $\mathcal{S}_n(n)$; in particular, rank $\mathcal{M}(n) \leq 2n+1$.*

Proof. The proof is by induction on $n \geq 2$. For $n = 2$ the statement is clearly true, so assume it holds for $n = k$ (≥ 2). Suppose $\mathcal{M}(k+1)$ is positive and recursively generated, with $YX = 1$ in $\mathcal{C}_{\mathcal{M}(k+1)}$. Let $i, j \geq 0$, $i+j \leq k$. By the induction hypothesis, each column of the form $[Y^i X^j]_{m(k)}$ is in $\mathcal{S}_k(k)$, and since $\mathcal{M}(k+1) \geq 0$, the Extension Principle [Fia1, Proposition 2.4] shows that $Y^i X^j \in \mathcal{S}_k(k+1)$ ($\subseteq \mathcal{S}_{k+1}(k+1)$). Since $X^{k+1}, Y^{k+1} \in \mathcal{S}_{k+1}(k+1)$, it now suffices to consider a column in $\mathcal{M}(k+1)$ of the form $Y^{k+1-j} X^j$, with $1 \leq j \leq k$. Let $q(x, y) := yx - 1$ and let $p_{ij}(x, y) := y^i x^j$, so that $Y^{k+1-j} X^j = p_{k+1-j,j}(X, Y)$.

Also, let $r_{ij}(x,y) := y^{k+1-j}x^j - y^{k-j}x^{j-1}$. Now $r_{ij}(x,y) = y^{k-j}x^{j-1}(yx - 1) = p_{k-j,j-1}(x,y)q(x,y)$; since $\mathcal{M}(k+1)$ is recursively generated and $q(X,Y) = 0$, it follows that $r_{ij}(X,Y) = 0$, that is, $Y^{k+1-j}X^j = Y^{k-j}X^{j-1}$ in $\mathcal{C}_{\mathcal{M}(k+1)}$. By induction, $[Y^{k-j}X^{j-1}]_{m(k)} \in \mathcal{S}_k(k)$, and since $\mathcal{M}(k+1) \geq 0$, it follows as above that $Y^{k-j}X^{j-1} \in \mathcal{S}_{k+1}(k)$. Thus $Y^{k+1-j}X^j \ (= Y^{k-j}X^{j-1}) \in \mathcal{S}_{k+1}(k) \subseteq \mathcal{S}_{k+1}(k+1)$, as desired. $\qquad\square$

We next present two auxiliary results that will be used frequently in the sequel. Recall that for $i+j, k+\ell \leq n$, $\langle Y^iX^j, Y^kX^\ell \rangle$ denotes the entry of $\mathcal{M}(n)$ in row Y^kX^ℓ, column Y^iX^j, namely $\beta_{i+k,j+\ell}$. We extend this inner product notation from monomials to polynomials as follows. For $p \equiv \sum_{0 \leq i+j \leq n} a_{ij}y^ix^j$ and $q \equiv \sum_{0 \leq k+\ell \leq n} b_{k\ell}y^kx^\ell$, we define $\langle p(X,Y), q(X,Y) \rangle := \sum_{0 \leq i+j, k+\ell \leq n} a_{ij}b_{k\ell}\beta_{i+k,j+\ell}$. Further, if $\deg p + \deg p'$, $\deg q + \deg q' \leq n$, by $\langle p(X,Y)p'(X,Y), q(X,Y)q'(X,Y) \rangle$ we mean $\langle (pp')(X,Y), (qq')(X,Y) \rangle$. The following result follows directly from the preceding definitions.

Lemma 2.3. *(i) For $p, q \in \mathbb{R}_n[x,y]$,*

$$\langle p(X,Y), q(X,Y) \rangle = \langle q(X,Y), p(X,Y) \rangle.$$

(ii) For $p, q \in \mathbb{R}_n[x,y]$, $i, j \geq 0$, $i + j \leq n$, and $\deg p, \deg q \leq n - (i+j)$,

$$\langle p(X,Y)Y^jX^i, q(X,Y) \rangle = \langle p(X,Y), q(X,Y)Y^jX^i \rangle.$$

(iii) If $p, q, r \in \mathbb{R}_n[x,y]$ with $p(X,Y) = q(X,Y)$ in $\mathcal{C}_{\mathcal{M}(n)}$, then $\langle r(X,Y), p(X,Y) \rangle = \langle r(X,Y), q(X,Y) \rangle$.

Lemma 2.4. *Let $\mathcal{M}(n)$ be positive, recursively generated, with $YX = 1$, and assume $p, q \in \mathbb{R}_{n-1}[x,y]$. Then*

$$\langle p(X,Y), q(X,Y) \rangle = \langle Yp(X,Y), Xq(X,Y) \rangle \tag{2.1}$$

$$= \langle Xp(X,Y), Yq(X,Y) \rangle. \tag{2.2}$$

Proof. The definition of $\langle p(X,Y), q(X,Y) \rangle$ implies that, without loss of generality, we can assume that $p(X,Y) = Y^iX^j$ and $q(X,Y) = Y^kX^\ell$. Assume first that $k \geq 1$. We have

$$\begin{aligned}
\langle Y^iX^j, Y^kX^\ell \rangle &= \langle Y^{i+1}X^j, Y^{k-1}X^\ell \rangle \quad \text{(by Lemma 2.3(ii))}\\
&= \langle Y^{i+1}X^j, Y^{k-1}X^\ell YX \rangle \quad \text{(by Lemma 2.3(iii), using } YX = 1)\\
&= \langle Y^{i+1}X^j, Y^kX^{\ell+1} \rangle.
\end{aligned}$$

If $k = 0$ and $j \geq 1$, we have

$$\begin{aligned}
\langle Y^iX^j, X^\ell \rangle &= \langle Y^iX^{j-1}, X^{\ell+1} \rangle \quad \text{(by Lemma 2.3(ii))}\\
&= \langle Y^iX^{j-1}YX, X^{\ell+1} \rangle \quad \text{(by Lemma 2.3(iii), using } YX = 1)\\
&= \langle Y^{i+1}X^j, X^{\ell+1} \rangle.
\end{aligned}$$

If $k = j = 0$, we have $p(X, Y) = Y^i$, $q(X, Y) = X^\ell$, so we need to prove that $\langle Y^i, X^\ell \rangle = \langle Y^{i+1}, X^{\ell+1} \rangle$. If $i \geq 1$, we have

$$\langle Y^i, X^\ell \rangle = \langle Y^{i-1}, YX^\ell \rangle \quad \text{(by Lemma 2.3(ii))}$$
$$= \langle Y^i X, YX^\ell \rangle \quad \text{(by Lemma 2.3(i),(iii))}$$
$$= \beta_{i+1,\ell+1} = \langle Y^{i+1}, X^{\ell+1} \rangle.$$

If $i = 0$, then

$$\langle Y^i, X^\ell \rangle = \langle 1, X^\ell \rangle = \langle YX, X^\ell \rangle \quad \text{(by Lemma 2.3(iii))}$$
$$\langle Y, X^{\ell+1} \rangle \quad \text{(by Lemma 2.3(ii))}.$$

We have now completed the proof of (2.1); the validity of (2.2) is a straightforward consequence of (2.1) and Lemma 2.3(i). $\qquad \square$

We next divide the proof of Theorem 2.1 into four cases, based on possible dependence relations among the elements of $\mathcal{S}_n(n)$. Section 5 contains examples illustrating these cases. In proving each case, we ultimately obtain some flat moment matrix extension $\mathcal{M}$; the existence of a corresponding rank $\mathcal{M}$-atomic representing measure μ supported in $yx = 1$ then follows immediately from (1.8) and (1.5); for this reason, and to simplify the statement of each case, we address only the matrix extension, not the representing measure. In the sequel, unless otherwise noted, we are always assuming that $\mathcal{M}(n)$ is positive, recursively generated, $\{1, X, Y\}$ is linearly independent, $YX = 1$, and rank $\mathcal{M}(n) \leq \operatorname{card} \mathcal{V}(\beta)$.

Proposition 2.5. *(Case I: For some k, $2 \leq k \leq n$, $\mathcal{S}_n(k-1)$ is linearly independent and $X^k \in lin.span\, \mathcal{S}_n(k-1)$) Assume that $\mathcal{M}(n)(\beta)$ is positive, recursively generated, satisfies $YX = 1$, and rank $\mathcal{M}(n) \leq \operatorname{card} \mathcal{V}(\beta)$. In $\mathcal{S}_n(n)$, assume that the first dependence relation occurs at X^k, with $2 \leq k \leq n$. Then $\mathcal{M}(n)$ is flat and, a fortiori, it admits a unique flat extension $\mathcal{M}(n+1)$.*

Proof. Write $X^k = p_{k-1}(X) + q_{k-1}(Y)$, where $\deg p_{k-1}, \deg q_{k-1} \leq k-1$. It follows that $\mathcal{V}(\beta) \subseteq (yx = 1) \bigcap (p_{k-1}(x) + q_{k-1}(y) = x^k) \subseteq (yx = 1) \bigcap (p_{k-1}(x) + q_{k-1}(\frac{1}{x}) = x^k)$. Since $p_{k-1}(x) + q_{k-1}(\frac{1}{x}) = x^k$ leads to a polynomial equation in x of degree at most $2k - 1$, it follows that $\operatorname{card} \mathcal{V}(\beta) \leq 2k - 1$, so rank $\mathcal{M}(n) \leq 2k - 1$. Then $\mathcal{S}_n(k-1) \equiv \{1, X, Y, X^2, Y^2, ..., X^{k-1}, Y^{k-1}\}$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$, whence $\mathcal{M}(n)$ is flat. $\qquad \square$

Proposition 2.6. *(Case II: For some k, $2 \leq k < n$, $\mathcal{S}_n(k-1) \bigcup \{X^k\}$ is linearly independent, and $Y^k \in lin.span\, (\mathcal{S}_n(k-1) \bigcup \{X^k\})$) Assume that $\mathcal{M}(n)(\beta)$ is positive, recursively generated, and satisfies $YX = 1$. In $\mathcal{S}_n(n)$, assume that the first dependence relation occurs at Y^k, with $1 \leq k < n$. Then $\mathcal{M}(n)$ is flat, and thus admits a unique flat extension $\mathcal{M}(n+1)$.*

Proof. Write $Y^k = p_k(X) + q_{k-1}(Y)$, where $\deg p_k \leq k$ and $\deg q_{k-1} \leq k - 1$. Since Y^k corresponds to a monomial of degree at most $n - 1$, and since $YX = 1$ and $\mathcal{M}(n)$ is recursively generated, we must have

$$Y^{k-1} = XY^k = Xp_k(X) + Xq_{k-1}(Y). \tag{2.3}$$

Since $\mathcal{M}(n)$ is recursively generated and $YX = 1$, $Xq_{k-1}(Y)$ is clearly a linear combination of columns corresponding to monomials of degree at most $k-2$. Let a_k be the coefficient of X^k in p_k. If $a_k = 0$, it follows from (2.3) that $\mathcal{S}_n(k-1)\bigcup\{X^k\}$ is linearly dependent, a contradiction. Thus, we must have $a_k \neq 0$, whence (2.3) implies that X^{k+1} is a linear combination of previous columns. Moreover, $Y^{k+1} = Yp_k(X) + Yq_{k-1}(Y)$, and $Yp_k(X)$ has degree $k-1$ in X, so $\mathcal{M}(k+1)$ is flat. It now follows from the Extension Principle [Fia1] and recursiveness that $\mathcal{M}(n)$ is flat, i.e., $\mathcal{M}(n)$ is a flat extension of $\mathcal{M}(k)$. $\qquad\square$

Proposition 2.7. *(Case III: The first dependence relation occurs at Y^n)* *Assume that $\mathcal{M}(n)(\beta)$ is positive, recursively generated, satisfies $YX = 1$, and* rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\beta)$. *In $\mathcal{S}_n(n)$, assume that Y^n is the location of the first dependence relation. Then $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$.*

The proof of Proposition 2.7 will require several preliminary results (Lemmas 2.8-2.11 below). Under the hypotheses of Proposition 2.7, write

$$Y^n = a_n X^n + p_{n-1}(X) + q_{n-1}(Y), \tag{2.4}$$

with $\deg p_{n-1}, \deg q_{n-1} \leq n-1$. We claim that $a_n \neq 0$. Assume instead that $a_n = 0$, i.e., $Y^n = p_{n-1}(X)+q_{n-1}(Y)$. Then $\mathcal{V}(\beta) \subseteq (yx = 1)\bigcap(p_{n-1}(x)+q_{n-1}(y) = y^n)$ $\subseteq (yx = 1)\bigcap(p_{n-1}(\frac{1}{y}) + q_{n-1}(y) = y^n)$. Since $p_{n-1}(\frac{1}{y}) + q_{n-1}(y) = y^n$ leads to a polynomial equation in y of degree at most $2n - 1$, it follows that card $\mathcal{V}(\beta) \leq 2n-1$, so rank $\mathcal{M}(n) \leq 2n-1$. Then $\mathcal{S}_n(n-1) \equiv \{1, X, Y, X^2, Y^2, ..., X^{n-1}, Y^{n-1}\}$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$, whence X^n is a linear combination of the columns in $\mathcal{S}_n(n-1)$, a contradiction. Thus, $a_n \neq 0$, so in particular

$$X^n = \frac{1}{a_n}[Y^n - p_{n-1}(X) - q_{n-1}(Y)]. \tag{2.5}$$

To build a flat extension $\mathcal{M}(n+1) \equiv \begin{pmatrix} \mathcal{M}(n) & B(n+1) \\ B(n+1)^* & C(n+1) \end{pmatrix}$, we define the middle n columns of a prospective block $B \equiv B(n+1)$ by exploiting recursiveness and the relation $YX = 1$, as follows:

$$YX^n := X^{n-1}; \ Y^2X^{n-1} := YX^{n-2}; \ ..., \ Y^nX := Y^{n-1}. \tag{2.6}$$

Also, motivated by (2.5) and, respectively, (2.4), we let

$$X^{n+1} := \frac{1}{a_n}[Y^{n-1} - Xp_{n-1}(X) - Xq_{n-1}(Y)], \tag{2.7}$$

and

$$Y^{n+1} := a_n X^{n-1} + Yp_{n-1}(X) + Yq_{n-1}(Y). \tag{2.8}$$

(The expressions $Y^{n-1} - Xp_{n-1}(X) - Xq_{n-1}(Y)$ and $a_n X^{n-1} + Yp_{n-1}(X) + Yq_{n-1}(Y)$ are shorthand notation for $(y^{n-1} - xp_{n-1}(x) - xq_{n-1}(y))(X,Y)$ and $(a_n x^{n-1} + yp_{n-1}(x) + yq_{n-1}(y))(X,Y)$ in $\mathcal{C}_{\mathcal{M}(n)}$, respectively. Observe that these defining relations are all required if one is to obtain a positive recursively generated moment matrix extension $\mathcal{M}(n + 1)$.) Since the columns defined by (2.6) - (2.8) belong to $\mathcal{C}_{\mathcal{M}(n)}$, we have $B = \mathcal{M}(n)W$ for some matrix W. Thus, a

flat extension $M := [\mathcal{M}(n); B]$ is uniquely determined by defining the C-block as $C := W^*\mathcal{M}(n)W$ (cf. Section 1). To complete the proof that M is a moment matrix $\mathcal{M}(n+1)$, it suffices to show that block B is of the form $(B_{i,n+1})_{i=0}^{n}$ and that block C is of the form $B_{n+1,n+1}$. To this end, we require some additional notation and several preliminary results.

We next extend the notation $\langle p(X,Y), q(X,Y)\rangle$ to the case when $\deg p = n + 1$, $\deg q \le n$. Indeed, using the definitions of the columns of B, for $i, j \ge 0$, $i + j = n + 1$, there exists $p_{ij} \in \mathbb{R}_n[x, y]$ with $Y^i X^j = p_{ij}(X, Y)$, and we define

$$\langle Y^i X^j, q(X,Y)\rangle := \langle p_{ij}(X,Y), q(X,Y)\rangle .$$

Now if $p(x, y) \equiv \sum_{0 \le k+\ell \le n+1} a_{k\ell} x^\ell y^k$, we define

$$\langle p(X,Y), q(X,Y)\rangle := \sum_{0 \le k+\ell \le n+1} a_{k\ell} \langle Y^k X^\ell, q(X,Y)\rangle .$$

It is easy to check that Lemma 2.3(iii) holds with $\deg r = n + 1$.

Lemma 2.8. *Under the hypotheses of Proposition 2.7, assume $i, j \ge 0$, with $i + j = n + 1$, and $r, s \ge 1$, with $r + s \le n$. Then*

$$\langle Y^i X^j, Y^r X^s\rangle = \langle Y^i X^j, Y^{r-1} X^{s-1}\rangle . \tag{2.9}$$

Proof. Fix i and j with $i + j = n + 1$. We know from (2.6) - (2.8) that there exists a polynomial $p \in \mathbb{R}_n[x, y]$ such that $Y^i X^j = p(X, Y) \equiv \sum_{k+\ell \le n} a_{k,\ell} Y^k X^\ell$. Then

$$\langle Y^i X^j, Y^r X^s\rangle = \sum_{k+\ell \le n} a_{k,\ell} \langle Y^k X^\ell, Y^r X^s\rangle$$

$$= \sum_{k+\ell \le n} a_{k,\ell} \langle Y^r X^s, Y^k X^\ell\rangle \quad \text{(because } \mathcal{M}(n) \text{ is self-adjoint)}$$

$$= \sum_{k+\ell \le n} a_{k,\ell} \langle Y^{r-1} X^{s-1}, Y^k X^\ell\rangle \quad \text{(using } YX = 1 \text{ and recursiveness)}$$

$$= \sum_{k+\ell \le n} a_{k,\ell} \langle Y^k X^\ell, Y^{r-1} X^{s-1}\rangle$$

(using again the self-adjointness of $\mathcal{M}(n)$)

$$= \langle Y^i X^j, Y^{r-1} X^{s-1}\rangle ,$$

as desired. □

The next result provides a reduction for the proof that $B(n + 1)$ has the Hankel property.

Lemma 2.9. *Under the hypotheses of Proposition 2.7, assume $i + j = n + 1$, with $j \ge 1$, $i \ge 0$, and assume that the Hankel property*

$$\langle Y^i X^j, Y^r X^s\rangle = \langle Y^{i+1} X^{j-1}, Y^{r-1} X^{s+1}\rangle \tag{2.10}$$

holds with $1 \le r \le n$ and $s = 0$. Then (2.10) holds for all r and s such that $1 \le r + s \le n$, $r \ge 1$, $s \ge 0$.

Proof. Fix i and j with $i + j = n + 1$. We use induction on $t := r + s$, where $1 \le r + s \le n$, $r \ge 1$, $s \ge 0$. For $t = 1$ the result follows from the hypothesis, since $r = 1$, $s = 0$. Assume now that $t = 2$. By hypothesis, we may assume $r = s = 1$, so we consider the equation

$$\langle Y^i X^j, YX \rangle = \langle Y^{i+1} X^{j-1}, X^2 \rangle, \qquad (2.11)$$

with $j \ge 1$, $i \ge 0$, $i + j = n + 1$. Since $YX = 1$, the left-hand side of (2.11) equals $\langle Y^i X^j, 1 \rangle$ by Lemma 2.8. For $j \ge 2$ and $i \ge 1$, the right-hand side of (2.11) equals $\langle Y^i X^{j-2}, X^2 \rangle$ (by (2.6)), which in turn equals $\langle Y^i XX^{j-2}, X \rangle = \langle Y^{i-1} X^{j-2}, X \rangle = \langle Y^{i-1} X^{j-1}, 1 \rangle = \langle Y^i X^j, 1 \rangle$ (by (2.6) for the last step). When $j \ge 2$ and $i = 0$ (which then implies $j = n + 1$), we have

$$\langle X^{n+1}, YX \rangle = \left\langle \frac{1}{a_n} [Y^{n-1} - X p_{n-1}(X) - X q_{n-1}(Y)], YX \right\rangle \text{ (by (2.7)}$$

$$= \frac{1}{a_n} [\langle Y^n, X \rangle - \langle p_{n-1}(X), XYX \rangle - \langle q_{n-1}(Y), XYX \rangle]$$

(by Lemma 2.3(ii) for the first term and Lemma 2.4 for the

last two terms)

$$= \left\langle \frac{1}{a_n} [Y^n - p_{n-1}(X) - q_{n-1}(Y)], X \right\rangle \text{ (by Lemma 2.3(iii))}$$

$$= \langle X^n, X \rangle \text{ (by (2.5))}$$

$$= \langle X^{n-1}, X^2 \rangle$$

$$= \langle YX^n, X^2 \rangle \text{ (by 2.6)).}$$

When $j = 1$ (so that $i = n$), the right-hand side of (2.11) is

$$\langle Y^{n+1}, X^2 \rangle = \langle a_n X^{n-1} + Y p_{n-1}(X) + Y q_{n-1}(Y), X^2 \rangle \text{ (by(2.8))}$$

$$= \langle a_n X^n + p_{n-1}(X) + q_{n-1}(Y), X \rangle$$

(using Lemma 2.3(ii) and Lemma 2.4, as above)

$$= \langle Y^n, X \rangle \text{ (by (2.4))}$$

$$= \langle Y^{n-1}, 1 \rangle \text{ (again using Lemma 2.4).}$$

On the other hand, the left-hand side of (2.11) is

$$\langle Y^n X, YX \rangle = \langle Y^{n-1}, YX \rangle \text{ (by(2.6))}$$

$$= \langle Y^{n-1}, 1 \rangle \text{ (by Lemma 2.3(iii).}$$

This completes the case when $t = 2$.

Assume now that (2.10) is true for $t \le u$, with $u \ge 2$, and consider the case $t = u + 1$. Thus $r + s = u + 1$ ($\le n$), and we may assume $r, s \ge 1$. When $r \ge 2$,

$$\langle Y^i X^j, Y^r X^s \rangle = \langle Y^i X^j, Y^{r-1} X^{s-1} \rangle \text{ (by Lemma 2.8)}$$

$$= \langle Y^{i+1} X^{j-1}, Y^{r-2} X^s \rangle \text{ (by the inductive step)}$$

$$= \langle Y^{i+1} X^{j-1}, Y^{r-1} X^{s+1} \rangle \text{ (by Lemma 2.8),}$$

as desired. When $r = 1$ and $j \geq 2$, we have $s \leq n - r = n - 1$, and we consider three subcases.

Subcase 1. For $j = 2$, $i = n - 1$,

$$
\begin{aligned}
\langle Y^{n-1}X^2, YX^s \rangle &= \langle Y^{n-2}X, YX^s \rangle \quad \text{(by (2.6))} \\
&= \langle Y^{n-2}X, X^{s-1} \rangle \quad \text{(by Lemma 2.3(iii))} \\
&= \langle Y^{n-2}, X^s \rangle \\
&= \langle Y^{n-1}, X^{s+1} \rangle \quad \text{(by Lemma 2.4, since } s \leq n - 1) \\
&= \langle Y^n X, X^{s+1} \rangle \quad \text{(by (2.6))}.
\end{aligned}
$$

Subcase 2. For $j \geq 3$, $i \geq 1$,

$$
\begin{aligned}
\langle Y^i X^j, YX^s \rangle &= \langle Y^{i-1}X^{j-1}, YX^s \rangle \quad \text{(by (2.6))} \\
&= \langle Y^{i-1}X^{j-1}, X^{s-1} \rangle \quad \text{(by Lemma 2.3(iii))} \\
&= \langle Y^{i-1}X^{j-3}, X^{s+1} \rangle = \langle Y^i X^{j-2}, X^{s+1} \rangle \quad \text{(since } YX = 1 \text{ in } \mathcal{M}(n)) \\
&= \langle Y^{i+1}X^{j-1}, X^{s+1} \rangle \quad \text{(by (2.6))}.
\end{aligned}
$$

Subcase 3. For $j = n + 1$, $i = 0$,

$$
\begin{aligned}
\langle X^{n+1}, YX^s \rangle &= \left\langle \frac{1}{a_n}[Y^{n-1} - Xp_{n-1}(X) - Xq_{n-1}(Y)], YX^s \right\rangle \\
&= \left\langle \frac{1}{a_n}[Y^n - p_{n-1}(X) - q_{n-1}(Y)], X^s \right\rangle \quad \text{(by Lemma 2.4)} \\
&= \langle X^n, X^s \rangle = \langle X^{n-1}, X^{s+1} \rangle \\
&= \langle YX^n, X^{s+1} \rangle \quad \text{(by (2.6))}.
\end{aligned}
$$

Finally, when $r = 1$ and $j = 1$, we have $i = n$ and $s \leq n - 1$, so

$$
\begin{aligned}
\langle Y^n X, YX^s \rangle &= \langle Y^{n-1}, YX^s \rangle \quad \text{(by (2.6))} \\
&= \langle Y^n, X^s \rangle = \langle a_n X^n + p_{n-1}(X) + q_{n-1}(Y), X^s \rangle \quad \text{(by (2.4))} \\
&= \langle a_n X^{n-1} + Yp_{n-1}(X) + Yq_{n-1}(Y), X^{s+1} \rangle \\
&\quad \text{(by Lemma 2.3(ii) and Lemma 2.4, as above)} \\
&= \langle Y^{n+1}, X^{s+1} \rangle.
\end{aligned}
$$

$\square$

Recall from (2.6) that columns $YX^n, ..., Y^n X$ are taken as a block from consecutive columns of degree $n - 1$ in $\mathcal{M}(n)$, so these columns satisfy the Hankel property. Thus, in view of Lemma 2.9, the next two results complete the proof that $B(n + 1)$ has the Hankel property.

Lemma 2.10. *For* $k = 1, ..., n$,

$$
\langle X^{n+1}, Y^k \rangle = \langle YX^n, Y^{k-1}X \rangle. \tag{2.12}
$$

Proof. We have

$$\left\langle X^{n+1}, Y^k \right\rangle = \left\langle \frac{1}{a_n}[Y^{n-1} - Xp_{n-1}(X) - Xq_{n-1}(Y)], Y^k \right\rangle \quad \text{(by (2.7))}$$

$$= \left\langle \frac{1}{a_n}[Y^n - p_{n-1}(X) - q_{n-1}(Y)], Y^{k-1} \right\rangle$$

(by Lemma 2.3(ii) and Lemma 2.4, as above)

$$= \left\langle X^n, Y^{k-1} \right\rangle = \left\langle X^{n-1}, Y^{k-1}X \right\rangle$$

$$= \left\langle YX^n, Y^{k-1}X \right\rangle \quad \text{(by (2.6))}.$$

$\square$

Lemma 2.11. *For* $k = 1, ..., n,$

$$\left\langle Y^n X, Y^k \right\rangle = \left\langle Y^{n+1}, Y^{k-1}X \right\rangle.$$

Proof. We have

$$\left\langle Y^n X, Y^k \right\rangle = \left\langle Y^{n-1}, Y^k \right\rangle \quad \text{(by (2.6)}$$

$$= \left\langle Y^n, Y^{k-1} \right\rangle = \left\langle a_n X^n + p_{n-1}(X) + q_{n-1}(Y), Y^{k-1} \right\rangle \quad \text{(by (2.4))}$$

$$= \left\langle a_n X^{n-1} + Yp_{n-1}(X) + Yq_{n-1}(Y), Y^{k-1}X \right\rangle$$

(by Lemma 2.4 for the last two terms)

$$= \left\langle Y^{n+1}, Y^{k-1}X \right\rangle \quad \text{(by (2.8))}$$

$\square$

The proof that block B is of the form $\{B_{i,n+1}\}_{i=0}^n$ is now complete. To finish the proof of Proposition 2.7 it now suffices to show that $C := W^*\mathcal{M}(n)W$ is Hankel. To do this, observe that in the C block of $\mathcal{M} := [\mathcal{M}(n); B] = \begin{pmatrix} \mathcal{M}(n) & B \\ B^* & C \end{pmatrix}$, we need to compute inner products of the form $\left\langle Y^i X^j, Y^k X^\ell \right\rangle$ $(i + j = k + \ell = n + 1)$. For this, we require an auxiliary lemma. For $i + j = k + \ell = n + 1$, by $\left\langle Y^i X^j, Y^k X^\ell \right\rangle$ we mean, as usual, the entry in row $Y^k X^\ell$ of column $Y^i X^j$; by self-adjointness of $\mathcal{M}(n+1)$, we have $\left\langle Y^i X^j, Y^k X^\ell \right\rangle = \left\langle Y^k X^\ell, Y^i X^j \right\rangle$. Now suppose $Y^i X^j = p(X, Y)$ and $Y^k X^\ell = q(X, Y)$, where $p(x, y) \equiv \sum_{0 \le r+s \le n} a_{rs} y^r x^s$ and $q(x, y) \equiv \sum_{0 \le t+u \le n} b_{tu} y^t x^u$; we define $\langle p(X, Y), q(X, Y) \rangle :=$ $\sum_{0 \le r+s, t+u \le n} a_{rs} \overline{b_{tu}} \left\langle \overline{Y}^r X^s, Y^t X^u \right\rangle$.

Lemma 2.12. *For* $i + j, k + \ell = n + 1$, $\left\langle Y^i X^j, Y^k X^\ell \right\rangle = \langle p(X, Y), q(X, Y) \rangle.$

Proof.

$$\langle p(X,Y), q(X,Y) \rangle \equiv \sum_{0 \leq r+s, t+u \leq n} a_{rs} b_{tu} \left\langle Y^r X^s, Y^t X^u \right\rangle$$

$$= \sum_{0 \leq t+u \leq n} b_{tu} \left\langle \sum_{0 \leq r+s \leq n} a_{rs} Y^r X^s, Y^t X^u \right\rangle$$

$$= \sum_{0 \leq t+u \leq n} b_{tu} \left\langle p(X,Y), Y^t X^u \right\rangle = \sum_{0 \leq t+u \leq n} b_{tu} \left\langle Y^i X^j, Y^t X^u \right\rangle$$

$$= \sum_{0 \leq t+u \leq n} b_{tu} \left\langle Y^t X^u, Y^i X^j \right\rangle \quad (\text{since } \mathcal{M} = \mathcal{M}^t)$$

$$= \left\langle \sum_{0 \leq t+u \leq n} b_{tu} Y^t X^u, Y^i X^j \right\rangle = \left\langle q(X,Y), Y^i X^j \right\rangle$$

$$= \left\langle Y^k X^\ell, Y^i X^j \right\rangle = \left\langle Y^i X^j, Y^k X^\ell \right\rangle \quad (\text{since } \mathcal{M} = \mathcal{M}^t).$$

$\square$

Proof of Proposition **2.7**. Note that since M is a flat extension, dependence relations in the columns of $(\mathcal{M}(n) \quad B)$ extend to column relations in $(B^* \quad C)$. In particular, the middle n columns of C coincide with the columns of degree $n-1$ of B^*; since B has the Hankel property, so does B^*, and thus the middle n columns of C have the Hankel property. To verify that C is Hankel it now suffices to focus on the first two and the last two columns of C, namely X^{n+1} and YX^n, and $Y^n X$ and Y^{n+1}. Since C is self-adjoint, and the middle n columns have the Hankel property, to check that C is Hankel it only remains to show that $C_{n+2,1} = C_{n+1,2}$, i.e., $\left\langle X^{n+1}, Y^{n+1} \right\rangle = \left\langle YX^n, Y^n X \right\rangle$. Now, by (2.7), (2.8) and Lemma 2.12 we have

$$\left\langle X^{n+1}, Y^{n+1} \right\rangle$$

$$= \left\langle \frac{1}{a_n} \{Y^{n-1} - X[p_{n-1}(X) + q_{n-1}(Y)]\}, a_n X^{n-1} + Y[p_{n-1}(X) + q_{n-1}(Y)] \right\rangle$$

$$= \left\langle Y^{n-1}, X^{n-1} \right\rangle + \frac{1}{a_n} \left\langle Y^{n-1}, Y[p_{n-1}(X) + q_{n-1}(Y)] \right\rangle$$

$$- \left\langle X[p_{n-1}(X) + q_{n-1}(Y)], X^{n-1} \right\rangle$$

$$- \frac{1}{a_n} \left\langle X[p_{n-1}(X) + q_{n-1}(Y)], Y[p_{n-1}(X) + q_{n-1}(Y)] \right\rangle$$

$$= \left\langle Y^n, X^n \right\rangle + \frac{1}{a_n} \left\langle Y^n, p_{n-1}(X) + q_{n-1}(Y) \right\rangle - \left\langle p_{n-1}(X) + q_{n-1}(Y), X^n \right\rangle$$

$$- \frac{1}{a_n} \left\langle p_{n-1}(X) + q_{n-1}(Y)], p_{n-1}(X) + q_{n-1}(Y)] \right\rangle$$

(by Lemma 2.4 for the first and fourth terms, and Lemma 2.3(ii) for
the second and third terms)

$$= \left\langle \frac{1}{a_n}\{Y^n - [p_{n-1}(X) + q_{n-1}(Y)]\}, a_n X^n + p_{n-1}(X) + q_{n-1}(Y) \right\rangle$$
$$= \langle X^n, Y^n \rangle \quad \text{(by (2.5), (2.4), and Lemma 2.3(i),(iii))}$$
$$= \left\langle X^{n-1}, Y^{n-1} \right\rangle \quad \text{(by Lemma 2.4)}$$
$$= \langle Y X^n, Y^n X \rangle \quad \text{(by (2.6) and Lemma 2.12).}$$

This concludes the proof of Proposition 2.7. $\hfill\square$

Remark 2.13. It is important for the sequel to note that in the proof of Proposition 2.7, the variety condition $\operatorname{rank} \mathcal{M}(n) \leq \operatorname{card} \mathcal{V}(\mathcal{M}(n))$ was used only to show that $a_n \neq 0$ in (2.4). Thus, if $\mathcal{M}(n)$ is positive, recursively generated, satisfies $YX = 1$ in $\mathcal{C}_{\mathcal{M}(n)}$, and if the first dependence relation in $\mathcal{S}_n(n)$ is of the form (2.4) with $a_n \neq 0$, then we may conclude that $\mathcal{M}(n)$ has a flat extension $\mathcal{M}(n+1)$.

Proposition 2.14. *(Case IV:* $\operatorname{rank} \mathcal{M}(n) = 2n+1$) *Assume that $\mathcal{M}(n)$ is positive, recursively generated, and satisfies $YX = 1$. Assume also that $\mathcal{S}_n(n)$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$. Then $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$, or $\mathcal{M}(n)$ admits a positive, recursively generated extension $\mathcal{M}(n+1)$, with $\operatorname{rank} \mathcal{M}(n+1) = 2n+2$, and $\mathcal{M}(n+1)$ admits a flat extension $\mathcal{M}(n+2)$.*

Proof. Since $YX = 1$, and to guarantee that $\mathcal{M}(n+1)$ is recursively generated, we define the middle n columns of a proposed B block for $\mathcal{M}(n+1)$ as $[YX^n]_{m(n)} := X^{n-1} \in \mathcal{C}_{\mathcal{M}(n)}$, $\left[Y^2 X^{n-1}\right]_{m(n)} := YX^{n-2} \in \mathcal{C}_{\mathcal{M}(n)}$, ... , $[Y^n X]_{m(n)} := Y^{n-1} \in \mathcal{C}_{\mathcal{M}(n)}$. Moreover, if we wish to make $B_{n,n+1}$ Hankel, it is clear that all but the entry $\langle X^{n+1}, X^n \rangle$ in the column $\left[X^{n+1}\right]_{m(n)}$ must be given in terms of entries in $\mathcal{M}(n)$, and that all but the entry $\langle Y^{n+1}, Y^n \rangle$ in $\left[Y^{n+1}\right]_{m(n)}$ must be given in terms of entries in $\mathcal{M}(n)$. To handle the remaining entries we introduce two parameters p and q; concretely, for $i + j = 0, ..., n$,

$$\left\langle X^{n+1}, Y^i X^j \right\rangle := \begin{cases} \langle YX^n, Y^{i-1}X^{j+1} \rangle & (1 \leq i \leq n) \\ \beta_{0,n+j+1} & (i = 0, \ 0 \leq j \leq n-1) \\ p & (i = 0, \ j = n) \end{cases}, \qquad (2.13)$$

$$\left\langle Y^{n+1}, Y^i X^j \right\rangle := \begin{cases} \langle Y^n X, Y^{i+1}X^{j-1} \rangle & (1 \leq j \leq n) \\ \beta_{n+1+i,0} & (0 \leq i \leq n-1, j = 0) \\ q & (i = n, j = 0) \end{cases}. \qquad (2.14)$$

A positive extension $\mathcal{M}(n+1)$ entails $\operatorname{Ran} B \subseteq \operatorname{Ran} \mathcal{M}(n)$, so in particular we must show that $\left[X^{n+1}\right]_{m(n)}, \left[Y^{n+1}\right]_{m(n)} \in \operatorname{Ran} \mathcal{M}(n)$. To this end, note that since $N := [\mathcal{M}(n)]_{\mathcal{S}_n(n)} > 0$, there exist vectors $\mathbf{f}, \mathbf{g} \in \mathbb{R}^{2n+1}$ such that $N\mathbf{f} = \left[X^{n+1}\right]_{\mathcal{S}_n(n)}$ and $N\mathbf{g} = \left[Y^{n+1}\right]_{\mathcal{S}_n(n)}$. Let $\mathbf{F}, \mathbf{G} \in \mathbb{R}^{m(n)}$ be given by

$$\left\langle \mathbf{F}, Y^i X^j \right\rangle := \begin{cases} \langle \mathbf{f}, Y^i X^j \rangle & \text{if } Y^i X^j \in \mathcal{S}_n(n) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\left\langle \mathbf{G}, Y^i X^j \right\rangle := \begin{cases} \langle \mathbf{g}, Y^i X^j \rangle & \text{if } Y^i X^j \in \mathcal{S}_n(n) \\ 0 & \text{otherwise} \end{cases}.$$

We observe, for future reference, that since $\mathbf{f} = N^{-1} \left[X^{n+1} \right]_{\mathcal{S}_n(n)}$, $\mathbf{f}$ is linear in p (and independent of q), and so also is $\mathbf{F}$; similarly, $\mathbf{g}$ and $\mathbf{G}$ are linear in q and independent of p.

Claim. $\mathcal{M}(n)\mathbf{F} = \left[X^{n+1} \right]_{m(n)}$; equivalently,

$$\left\langle \mathcal{M}(n)\mathbf{F}, Y^i X^j \right\rangle = \left\langle \left[X^{n+1} \right]_{m(n)}, Y^i X^j \right\rangle$$

for each $(i,j) \in I_u := \{ (i,j) : i+j \leq n \text{ and } ((i = u \leq j \leq n) \text{ or } (j = u \leq i \leq n)) \}$, $u = 0, 1, ..., \left[\frac{n}{2} \right]$. Our proof of Claim 1 is by induction on u. For $u = 0$, we consider $Z \equiv Y^i$ or $Z \equiv X^j$ in $\mathcal{S}_n(n)$, so

$$\left\langle \mathcal{M}(n)\mathbf{F}, Z \right\rangle = \sum_{Y^k X^\ell \in \mathcal{C}_{\mathcal{M}(n)}} \left\langle Y^k X^\ell, Z \right\rangle \left\langle \mathbf{F}, Y^k X^\ell \right\rangle$$

$$= \sum_{Y^k X^\ell \in \mathcal{S}_n(n)} \left\langle Y^k X^\ell, Z \right\rangle \left\langle \mathbf{f}, Y^k X^\ell \right\rangle + \sum_{Y^k X^\ell \notin \mathcal{S}_n(n)} \left\langle Y^k X^\ell, Z \right\rangle \cdot 0$$

$$= \left\langle N\mathbf{f}, Z \right\rangle = \left\langle \left[X^{n+1} \right]_{m(n)}, Z \right\rangle,$$

as desired. We must now deal with rows of the form $Y^i X^j$ $(i, j \geq 1)$; that is, we must prove that $\left\langle \mathcal{M}(n)\mathbf{F}, Y^i X^j \right\rangle = \left\langle \left[X^{n+1} \right]_{m(n)}, Y^i X^j \right\rangle$ for $i, j \geq 1$ and $i + j \leq n$. Assume that the Claim is true for $u = k$ $(0 \leq k \leq \left[\frac{n}{2} \right] - 1)$, and consider $(i,j) \in I_{k+1}$. We have

$$\left\langle \left[X^{n+1} \right]_{m(n)}, Y^i X^j \right\rangle \equiv \left\langle X^{n+1}, Y^i X^j \right\rangle = \left\langle Y X^n, Y^{i-1} X^{j+1} \right\rangle \quad \text{(by (2.13)}$$

$$= \left\langle X^{n-1}, Y^{i-1} X^{j+1} \right\rangle.$$

On the other hand,

$$\left\langle \mathcal{M}(n)\mathbf{F}, Y^i X^j \right\rangle = \left\langle \mathcal{M}(n)\mathbf{F}, Y^{i-1} X^{j-1} \right\rangle \quad \text{(by Lemma 2.3(iii))}$$

$$= \left\langle \left[X^{n+1} \right]_{m(n)}, Y^{i-1} X^{j-1} \right\rangle \quad \text{(by the inductive step)}$$

$$= \left\langle X^{n+1}, Y^{i-1} X^{j-1} \right\rangle.$$

It thus suffices to prove that $\left\langle X^{n+1}, Y^{i-1} X^{j-1} \right\rangle = \left\langle X^{n-1}, Y^{i-1} X^{j+1} \right\rangle$. For $i = 1$,

$$\left\langle X^{n+1}, X^{j-1} \right\rangle = \beta_{0, n+j} \quad \text{(by (2.13))}$$

$$= \left\langle X^{n-1}, X^{j+1} \right\rangle,$$

and for $i > 1$,

$$\left\langle X^{n+1}, Y^{i-1} X^{j-1} \right\rangle = \left\langle Y X^n, Y^{i-2} X^j \right\rangle \quad \text{(by (2.13)}$$

$$= \left\langle X^{n-1}, Y^{i-2} X^j \right\rangle = \left\langle X^{n-1}, Y^{i-1} X^{j+1} \right\rangle \quad \text{(by Lemma 2.3(iii)}.$$

This completes the proof of the Claim. An entirely similar argument, using $\mathbf{g}$ instead of $\mathbf{f}$ and (2.14) instead of (2.13), shows that $\mathcal{M}(n)\mathbf{G} = \left[Y^{n+1} \right]_{m(n)}$. Moreover, by definition, $\left[Y^i X^j \right]_{m(n)} = Y^{i-1} X^{j-1} \in \mathcal{C}_{\mathcal{M}(n)}$ $(i + j = n + 1; i, j \geq 1)$,

so we now have Ran $B \subseteq$ Ran $\mathcal{M}(n)$; in particular, there exists W such that $\mathcal{M}(n)W = B$.

We note the following for future reference. From Lemma 2.2 and the fact that $\mathcal{M}(n) = \mathcal{M}(n)^t$, each row of $\mathcal{M}(n)$ coincides with a row indexed by an element of $\mathcal{S}_n(n)$. Since $B = \mathcal{M}(n)W$, it now follows that each row of $(\mathcal{M}(n)\ B)$ coincides with a row of $(\mathcal{M}(n)\ B)$ indexed by an element of $\mathcal{S}_n(n)$.

We now form the flat extension $\mathcal{M} := [\mathcal{M}(n); B] \equiv \begin{pmatrix} \mathcal{M}(n) & B \\ B^t & C \end{pmatrix}$, where $C := W^t \mathcal{M}(n)W$. Exactly as in the proof of Proposition 2.7, C is of the form

$$C \equiv \begin{pmatrix} \tau & \beta_{0,2n} & \cdots & \beta_{02} & \eta \\ \beta_{0,2n} & \beta_{0,2n-2} & \cdots & \beta_{00} & \beta_{20} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_{02} & \beta_{00} & \cdots & \beta_{2n-2,0} & \beta_{2n,0} \\ \eta & \beta_{20} & \cdots & \beta_{2n,0} & \rho \end{pmatrix},$$

where $C_{11} \equiv \tau := \left[X^{n+1}\right]^t_{\mathcal{S}_n(n)} N^{-1} \left[X^{n+1}\right]_{\mathcal{S}_n(n)}$ and $C_{1,n+2} = C_{n+2,1} \equiv \eta := \left[X^{n+1}\right]^t_{\mathcal{S}_n(n)} N^{-1} \left[Y^{n+1}\right]_{\mathcal{S}_n(n)}$. Thus, if $\eta = \beta_{00}$, then $\mathcal{M}$ is a flat moment matrix extension of the form $\mathcal{M}(n+1)$, and we are done.

Assume now that $\eta \neq \beta_{00}$. Let $u > \tau$ be arbitrary, and consider the moment matrix $\mathcal{M}' \equiv \mathcal{M}(n+1)'$ obtained from $\mathcal{M}$ by replacing τ by u and η by β_{00}. We partition $\mathcal{M}'$ as $\mathcal{M}' \equiv \begin{pmatrix} \tilde{\mathcal{M}} & \tilde{B} \\ \tilde{B}^t & \tilde{C} \end{pmatrix}$, where $\tilde{\mathcal{M}}$ is the compression of $\mathcal{M}'$ to rows and columns indexed by $\tilde{\mathcal{B}} := \{1, X, Y, X^2,\ YX, Y^2, ...,\ X^n, YX^{n-1}, ..., Y^{n-1}X, Y^n, X^{n+1}\} \subseteq \mathcal{C}_{\mathcal{M}'}$ (i.e., $\tilde{\mathcal{M}}$ is the extension of $\mathcal{M}(n)$ by row X^{n+1} and column X^{n+1} of $\mathcal{M}'$).

We claim that Ran $\tilde{B} \subseteq$ Ran $\tilde{\mathcal{M}}$. By the flat construction of $\mathcal{M}$, the "middle" columns of $\begin{pmatrix} B \\ C \end{pmatrix}$ are borrowed from columns of $\begin{pmatrix} \mathcal{M}(n) \\ B^t \end{pmatrix}$ of degree $n-1$, so, in particular, the columns of $\tilde{B}$ (except the rightmost column) are borrowed from columns in $\tilde{\mathcal{M}}$. To prove the claim, it thus suffices to show that $\left[Y^{n+1}\right]_{\tilde{\mathcal{B}}} \in$ Ran $\tilde{\mathcal{M}}$. Since $u > \tau$, $\tilde{\mathcal{S}} := \{1, X, Y, X^2, Y^2, ..., X^n, Y^n, X^{n+1}\}$ is a basis for $\mathcal{C}_{\tilde{\mathcal{M}}}$, and $[\tilde{\mathcal{M}}]_{\tilde{\mathcal{S}}}$ is positive and invertible. Thus there exist unique scalars $a_1, a_2, ..., a_{2n+2}$ such that, in $\mathcal{C}_{(\tilde{\mathcal{M}}\ \tilde{B})}$, we have

$$[Y^{n+1}]_{\tilde{\mathcal{S}}} = a_1[1]_{\tilde{\mathcal{S}}} + a_2[X]_{\tilde{\mathcal{S}}} + ... + a_{2n+1}[Y^n]_{\tilde{\mathcal{S}}} + a_{2n+2}[X^{n+1}]_{\tilde{\mathcal{S}}}.$$

From the first part of the proof (concerning block B), we know that each row of $(\tilde{\mathcal{M}}\ \tilde{B})$ coincides with a row indexed by an element of $\tilde{\mathcal{S}}$, so it now follows that, in $\mathcal{C}_{(\tilde{\mathcal{M}}\ \tilde{B})}$,

$$[Y^{n+1}]_{\tilde{\mathcal{B}}} = a_1[1]_{\tilde{\mathcal{B}}} + a_2[X]_{\tilde{\mathcal{B}}} + ... + a_{2n+1}[Y^n]_{\tilde{\mathcal{B}}} + a_{2n+2}[X^{n+1}]_{\tilde{\mathcal{B}}}, \qquad (2.15)$$

whence the claim is proved.

Since $\tilde{\mathcal{M}} \geq 0$ and Ran $\tilde{B} \subseteq$ Ran $\tilde{\mathcal{M}}$, we may construct the (positive) flat extension $\mathcal{M}^{\flat} := [\tilde{\mathcal{M}}; \tilde{B}] \equiv \begin{pmatrix} \tilde{\mathcal{M}} & \tilde{B} \\ \tilde{B}^t & D \end{pmatrix}$, which we may re-partition as the moment matrix $\mathcal{M}(n+1) = \begin{pmatrix} \mathcal{M}(n) & B \\ B^t & C^{\flat} \end{pmatrix}$, where $C^{\flat}$ is obtained from C by replacing τ by u, η by β_{00}, and ρ by some $\rho^{\flat}$ (determined by extending (2.15) to the full columns of $\mathcal{M}^{\flat}$).

Now $\mathcal{M}(n+1)$ is positive, recursively generated, satisfies $YX = 1$, and (by flatness of $\mathcal{M}^{\flat}$), rank $\mathcal{M}(n+1) = $ rank $\tilde{\mathcal{M}} = 1+$rank $\mathcal{M}(n)$. In $\mathcal{S}_{n+1}(n+1)$, the first dependence relation is of the form $Y^{n+1} = a_1 1 + a_2 X + ... + a_{2n+1} Y^n + a_{2n+2} X^{n+1}$, and we assert that $a_{2n+2} \neq 0$. Indeed, if $a_{2n+2} = 0$, then $[Y^{n+1}]_{\mathcal{S}_n(n)} = a_1 [1]_{\mathcal{S}_n(n)} + a_2 [X]_{\mathcal{S}_n(n)} + ... + a_{2n+1} [Y^n]_{\mathcal{S}_n(n)}$, whence $(a_1, ..., a_{2n+1})^t = N^{-1} [Y^{n+1}]_{\mathcal{S}_n(n)}$. Now we have

$$\begin{aligned} \beta_{00} &= \left\langle [Y^{n+1}]_{\tilde{\mathcal{B}}}, X^{n+1} \right\rangle \\ &= a_1 \left\langle [1]_{\tilde{\mathcal{B}}}, X^{n+1} \right\rangle + a_2 \left\langle [X]_{\tilde{\mathcal{B}}}, X^{n+1} \right\rangle + ... + a_{2n+1} \left\langle [Y^n]_{\tilde{\mathcal{B}}}, X^{n+1} \right\rangle \\ &= [X^{n+1}]^t_{\mathcal{S}_n(n)} \cdot (a_1, ..., a_{2n+1})^t \\ &= [X^{n+1}]^t_{\mathcal{S}_n(n)} N^{-1} [Y^{n+1}]_{\mathcal{S}_n(n)} = \eta, \end{aligned}$$

a contradiction. Since $a_{2n+2} \neq 0$, we may now proceed exactly as in the proof of Proposition 2.7 (beginning at (2.5) and replacing n by $n+1$) to conclude that $\mathcal{M}(n+1)$ admits a flat extension $\mathcal{M}(n+2)$ (cf. Remark 2.13). $\square$

Remark 2.15. Recall that $\mathbf{F}$ depends on p and is independent of q, while $\mathbf{G}$ depends on q and is independent of p. It follows that η is of the form $\eta \equiv \eta(p,q) = a + bp + cq + dpq$, where $a, b, c, d \in \mathbb{R}$ are independent of p and q. Thus, if b, c or d is nonzero, it is possible to choose p and q so that $\eta = \beta_{00}$, whence $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$ (and β admits a rank $\mathcal{M}(n)$-atomic representing measure). In [CuFi5, Proof of Proposition 5.3] we showed that this is the case in the quartic moment problem ($n = 2$), where we always have b or d nonzero. For $n > 2$, we do not know whether it is always the case that b, c or d is nonzero.

Proof of Theorem **2.1**. Straightforward from Propositions 2.5, 2.6, 2.7 and 2.14. $\square$

3. The Truncated Moment Problem on Degenerate Hyperbolas

In this section we prove Theorem 1.1 for degenerate hyperbolas. By Proposition 1.9, it suffices to consider the case $yx = 0$, and the necessity of the conditions in Theorem 1.1 is clear from Section 1. We establish sufficiency in the following result.

Theorem 3.1. *Let* $\beta \equiv \beta^{(2n)} : \beta_{00}, \beta_{01}, \beta_{10}, ..., \beta_{0,2n}, ..., \beta_{2n,0}$ *be a family of real numbers,* $\beta_{00} > 0$, *and let* $\mathcal{M}(n)$ *be the associated moment matrix. Assume that* $\mathcal{M}(n)$ *is positive, recursively generated, and satisfies* $YX = 0$ *and* rank $\mathcal{M}(n) \leq$

card $\mathcal{V}(\beta)$. Then rank $\mathcal{M}(n) \leq 2n + 1$. If rank $\mathcal{M}(n) \leq 2n$, then $\mathcal{M}(n)$ admits a flat extension (so β admits a rank $\mathcal{M}(n)$-atomic representing measure supported in $yx = 0$). If rank $\mathcal{M}(n) = 2n + 1$, then $\mathcal{M}(n)$ admits a positive, recursively generated extension $\mathcal{M}(n + 1)$, satisfying $2n + 1 \leq$ rank $\mathcal{M}(n + 1) \leq 2n + 2$, and $\mathcal{M}(n + 1)$ admits a flat extension $\mathcal{M}(n + 2)$ (so β admits a representing measure μ supported in $yx = 0$, with $2n + 1 \leq$ card supp $\mu \leq 2n + 2$).

By [CuFi2, Theorem 2.1], we know that a positive, recursively generated moment matrix $\mathcal{M}(n)$ admits flat extensions when $\{1, X, Y\}$ is linearly dependent in $\mathcal{C}_{\mathcal{M}(n)}$; in the sequel, we therefore assume that $\{1, X, Y\}$ is linearly independent. We begin with an elementary lemma based on recursiveness and $YX = 0$. For $1 \leq k \leq n$ let

$$\mathcal{S}_n(k) := \{1, X, Y, X^2, Y^2, ..., X^k, Y^k\} \subseteq \mathcal{C}_{\mathcal{M}(n)}.$$

Lemma 3.2. *For $n \geq 2$, let $\mathcal{M}(n)$ be positive and recursively generated, and assume that $YX = 0$. Then each nonzero column of $\mathcal{M}(n)$ is in $\mathcal{S}_n(n)$, and therefore rank $\mathcal{M}(n) \leq 2n + 1$.*

In the sequel we also require the following well-known result.

Lemma 3.3. *(Choleski's Algorithm* [Atk]*) Let A be a positive and invertible $d \times d$ matrix over $\mathbb{R}$, let $\mathbf{b}$ denote a column vector in $\mathbb{R}^d$, and let $c \in \mathbb{R}$. Then $\tilde{A} := \begin{pmatrix} A & \mathbf{b} \\ \mathbf{b}^t & c \end{pmatrix}$ is positive and invertible if and only if $\delta := c - \mathbf{b}^t A^{-1} \mathbf{b} > 0$. In this case,*

$$\tilde{A}^{-1} = \frac{1}{\delta} \begin{pmatrix} (\delta + A^{-1}\mathbf{b}\mathbf{b}^t)A^{-1} & -A^{-1}\mathbf{b} \\ -\mathbf{b}^t A^{-1} & 1 \end{pmatrix}.$$

We next divide the proof of Theorem 3.1 into three cases, based on possible dependence relations among the elements of $\mathcal{S}_n(n)$. Section 5 contains examples illustrating these cases. As in Section 2, in each case, once we establish a flat extension, the existence of the required representing measure μ, necessarily supported in $yx = 0$, always follows immediately from (1.5) and (1.8), so we will not repeat this argument in each case.

Proposition 3.4. *Suppose $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ is positive, recursively generated, card $\mathcal{V}(\mathcal{M}(n)) \geq$ rank $\mathcal{M}(n)$, and $YX = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$. Suppose there exists k, $1 < k \leq n$, such that $\mathcal{S}_n(k-1)$ is linearly independent and $X^k \in lin.span\, \mathcal{S}_n(k-1)$. Then $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n + 1)$ (and $\beta^{(2n)}$ admits a rank $\mathcal{M}(n)$-atomic representing measure).*

Proof. By hypothesis, we may write

$$X^k = a_0 1 + a_1 X + b_1 Y + ... + a_{k-1} X^{k-1} + b_{k-1} Y^{k-1} \quad (a_i, b_i \in \mathbb{R}). \tag{3.1}$$

Equation (3.1) implies that there are at most k points in $\mathcal{V}(\mathcal{M}(n))$ of the form $(x, 0)$. If, for some j, $b_j \neq 0$, then it follows from (3.1) that there are at most $k-1$ points in $\mathcal{V}(\mathcal{M}(n))$ of the form $(0, y)$. In this case, since $YX = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, it follows that $2k-1 = $ card $\mathcal{S}_n(k-1) \leq$ rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\mathcal{M}(n)) \leq 2k-1$, whence

rank $\mathcal{M}(n) = 2k-1$ and $Y^j \in \text{lin.span } \mathcal{S}_n(k-1)$ $(k \leq j \leq n)$. If $k = n$, this shows that $\mathcal{M}(n)$ is flat. If $k < n$, then also $X^j \in \text{lin.span } \mathcal{S}_n(k-1)$ $(k+1 \leq j \leq n)$, so again $\mathcal{M}(n)$ is flat. Thus, if some $b_j \neq 0$, then $\mathcal{M}(n)$ is flat and the result follows. We may thus assume that each $b_j = 0$, i.e.,

$$X^k = a_0 1 + a_1 X + ... + a_{k-1} X^{k-1}. \tag{3.2}$$

If $a_0 \neq 0$, there are no points in $\mathcal{V}(\mathcal{M}(n))$ of the form $(0, y)$. Thus, in this case, each point in the variety is of the form $(x, 0)$, and (3.2) implies that there can be at most k such points. Then $2k - 1 \leq \text{rank } \mathcal{M}(n) \leq \text{card } \mathcal{V}(\mathcal{M}(n)) \leq k$, a contradiction. We thus conclude that $a_0 = 0$, whence

$$X^k = a_1 X + ... + a_{k-1} X^{k-1}. \tag{3.3}$$

By recursiveness,

$$X^{k+i} = a_1 X^{i+1} + ... + a_{k-1} X^{k-1+i} \quad (1 \leq i \leq n - k). \tag{3.4}$$

Thus $\tilde{\mathcal{B}} := \{1, X, Y, ..., X^{k-1}, Y^{k-1}, Y^k, ..., Y^n\}$ spans $\mathcal{C}_{\mathcal{M}(n)}$. If $\tilde{\mathcal{B}}$ is linearly dependent, it follows readily from recursiveness and $YX = 0$ (as above) that $\mathcal{M}(n)$ is flat, so again there is a (unique, rank $\mathcal{M}(n)$-atomic) representing measure.

We may thus assume that $\tilde{\mathcal{B}}$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$. We will show that $\mathcal{M}(n)$ admits infinitely many flat extensions $\mathcal{M}(n+1)$ (each corresponding to a distinct rank $\mathcal{M}(n)$-atomic representing measure). To define

$$\mathcal{M}(n+1) \equiv \begin{pmatrix} \mathcal{M}(n) & B(n+1) \\ B(n+1)^t & C(n+1) \end{pmatrix},$$

which must be recursively generated (since $\mathcal{M}(n+1)$ would have a representing measure by (1.8)), we first use (3.3) and (3.4) to define X^{n+1} in $\mathcal{C}_{(\mathcal{M}(n) B(n+1))}$ by

$$X^{n+1} := a_1 X^{n+2-k} + ... + a_{k-1} X^n. \tag{3.5}$$

Clearly, $X^{n+1} \in \text{Ran } \mathcal{M}(n)$. To ensure the moment matrix structure of X^{n+1}, we must verify that

$$\langle X^{n+1}, Y^i X^j \rangle = \beta_{i, n+1+j} \quad (0 \leq i + j \leq n - 1). \tag{3.6}$$

If $i > 0$, then $\langle X^p, Y^i X^j \rangle = 0$ $(0 < p \leq n)$, so (3.5) implies $\langle X^{n+1}, Y^i X^j \rangle = 0 = \beta_{i, n+1+j}$ in this case. For $i = 0$,

$$\begin{aligned} \langle X^{n+1}, X^j \rangle &= \sum_{s=1}^{k-1} a_s \langle X^{n+1-k+s}, X^j \rangle = \sum_{s=1}^{k-1} a_s \langle X^{n-k+s}, X^{j+1} \rangle \\ &= \langle X^n, X^{j+1} \rangle \quad \text{(by (3.4))} \\ &= \beta_{0, n+1+j} \quad \text{(since } 0 \leq j \leq n - 1). \end{aligned}$$

We next define Y^{n+1} in $\mathcal{C}_{(\mathcal{M}(n) B(n+1))}$ consistent with moment matrix structure and $YX = 0$. Let $\mathcal{B} := \{1, X, ..., X^{k-1}\} \bigcup \{Y, ..., , Y^{k-1}, ..., Y^{n-1}\} \subseteq \mathcal{C}_{\mathcal{M}(n)};$

$[Y^{n+1}]_\mathcal{B}$ is uniquely determined by $\langle Y^{n+1}, X^j \rangle = 0$ $(1 \le j \le k-1)$ and $\langle Y^{n+1}, Y^i \rangle = \beta_{n+1+i,0}$ $(0 \le i \le n-1)$. Now

$$[Y^{n+1}]_{\tilde{\mathcal{B}}} = \left(\begin{array}{c} [Y^{n+1}]_\mathcal{B} \\ q \end{array} \right) \ (q \in \mathbb{R}), \tag{3.7}$$

and every other component of Y^{n+1} (besides those in $[Y^{n+1}]_{\tilde{\mathcal{B}}}$) must be zero (corresponding to $\langle Y^{n+1}, Y^r X^s \rangle$ for some $s > 0$).

We will show that for each $q \in \mathbb{R}$, $Y^{n+1} \in$ Ran $\mathcal{M}(n)$. Let $\mathcal{M} := [\mathcal{M}(n)]_\mathcal{B} > 0$. For each vector of the form $[v]_\mathcal{B}$, $\mathcal{M}^{-1}([v]_\mathcal{B}) \equiv (r_0, r_1, s_1, ..., r_{k-1}, s_{k-1}, ..., s_{n-1})$ uniquely satisfies

$$[v]_\mathcal{B} = r_0[1]_\mathcal{B} + r_1[X]_\mathcal{B} + s_1[Y]_\mathcal{B} + ... + r_{k-1}[X^{k-1}]_\mathcal{B} + s_{k-1}[Y^{k-1}]_\mathcal{B} + ... + s_{n-1}[Y^{n-1}]_\mathcal{B}. \tag{3.8}$$

We claim that

$$[X^{k+j}]_\mathcal{B}^t \mathcal{M}^{-1} [Y^n]_\mathcal{B} = 0 \ (0 \le j \le n-k). \tag{3.9}$$

Indeed, (3.3) and (3.4) together imply that X^{k+j} is a linear combination of X, $X^2, ... , X^{k-1}$, so in (3.8), all the coordinates of $\mathcal{M}^{-1}[X^{k+j}]_\mathcal{B}$ corresponding to $1, Y, ..., Y^{n-1}$ equal zero. By the $YX = 0$ structure of $\mathcal{M}(n)$, the only coordinates of $[Y^n]_\mathcal{B}$ that are possibly nonzero correspond to $1, Y, ..., Y^{n-1}$. These observations show that $[Y^n]_\mathcal{B}^t \mathcal{M}^{-1} [X^{k+j}]_\mathcal{B} = 0$, whence (3.9) follows. Similarly, the $YX = 0$ structure of Y^{n+1} implies that

$$[X^{k+j}]_\mathcal{B}^t \mathcal{M}^{-1} [Y^{n+1}]_\mathcal{B} = 0 \ (0 \le j \le n-k). \tag{3.10}$$

Since $\tilde{\mathcal{M}} := [\mathcal{M}(n)]_{\tilde{\mathcal{B}}} > 0$, we have $[Y^{n+1}]_{\tilde{\mathcal{B}}} \in$ Ran $\tilde{\mathcal{M}}$. To show that $Y^{n+1} \in$ Ran $\mathcal{M}(n)$, the $YX = 0$ structure of $\mathcal{M}(n)$ and of Y^{n+1} imply that it suffices to verify that

$$[X^{k+j}]_{\tilde{\mathcal{B}}}^t (\tilde{\mathcal{M}})^{-1} [Y^{n+1}]_{\tilde{\mathcal{B}}} = 0 \ (0 \le j \le n-k). \tag{3.11}$$

Now $\tilde{\mathcal{M}} = \left(\begin{array}{cc} \mathcal{M} & [Y^n]_\mathcal{B} \\ [Y^n]_\mathcal{B}^t & c \end{array} \right) \equiv \left(\begin{array}{cc} \mathcal{M} & \mathbf{b} \\ \mathbf{b}^t & c \end{array} \right)$, so

$$(\tilde{\mathcal{M}})^{-1} = \frac{1}{\delta} \left(\begin{array}{cc} (\delta + \mathcal{M}^{-1}\mathbf{b}\mathbf{b}^t)\mathcal{M}^{-1} & -\mathcal{M}^{-1}\mathbf{b} \\ -\mathbf{b}^t\mathcal{M}^{-1} & 1 \end{array} \right),$$

with $\delta := c - \mathbf{b}^t\mathcal{M}^{-1}\mathbf{b}$, by Lemma 3.3. We also have $[X^{k+j}]_{\tilde{\mathcal{B}}}^t = \left([X^{k+j}]_\mathcal{B}^t \ 0 \right)$ (since $\langle X^{k+j}, Y^n \rangle = 0$); thus, using (3.7), we have

$$[X^{k+j}]_{\tilde{\mathcal{B}}}^t \tilde{\mathcal{M}}^{-1} [Y^{n+1}]_{\tilde{\mathcal{B}}} = \frac{1}{\delta} [X^{k+j}]_\mathcal{B}^t ((\delta + \mathcal{M}^{-1}\mathbf{b}\mathbf{b}^t)\mathcal{M}^{-1} [Y^{n+1}]_\mathcal{B} - q\mathcal{M}^{-1}\mathbf{b})$$

$$= [X^{k+j}]_\mathcal{B}^t \mathcal{M}^{-1} [Y^{n+1}]_\mathcal{B}$$

$$+ \frac{1}{\delta} ([X^{k+j}]_\mathcal{B}^t \mathcal{M}^{-1}\mathbf{b})\mathbf{b}^t\mathcal{M}^{-1} [Y^{n+1}]_\mathcal{B}$$

$$- \frac{q}{\delta} [X^{k+j}]_\mathcal{B}^t \mathcal{M}^{-1}\mathbf{b}$$

$$= 0 \ \text{(by (3.9) and (3.10)).}$$

Thus $Y^{n+1} \in \operatorname{Ran} \mathcal{M}(n)$.

We now have columns X^{n+1} and Y^{n+1} for block $B(n+1)$, and to preserve the $YX = 0$ structure we set $Y^i X^j = 0$ $(i+j = n+1; i, j > 0)$. Thus Ran $B(n+1) \subseteq \operatorname{Ran} \mathcal{M}(n)$, and we consider the flat extension

$$\mathcal{M}\hat{} := [\mathcal{M}(n); B(n+1)] \equiv \begin{pmatrix} \mathcal{M}(n) & B(n+1) \\ B(n+1)^t & C \end{pmatrix}. \qquad (3.12)$$

To complete the proof, we will show that C has the form of a moment matrix block consistent with $YX = 0$. To do so, from (3.12) and the $YX = 0$ structures of $\mathcal{M}(n)$ and $B(n+1)$, it suffices to show that $\langle X^{n+1}, Y^{n+1} \rangle = 0$. Now,

$$\langle X^{n+1}, Y^{n+1} \rangle = \sum_{j=1}^{k-1} a_j \langle X^{n+1-k+j}, Y^{n+1} \rangle \quad \text{(by (3.5) and (3.12))}$$

$$= \sum_{j=1}^{k-1} a_j \langle Y^{n+1}, X^{n+1-k+j} \rangle \quad (\text{since } \mathcal{M}\hat{} = (\mathcal{M}\hat{})^t)$$

$$= 0 \quad (\text{since } Y^{n+1} \text{ in } B(n+1) \text{ has } YX = 0 \text{ structure}).$$

Thus $\mathcal{M}\hat{}$ is a flat moment matrix extension of $\mathcal{M}(n)$; the proof is complete. $\square$

We next consider the case when the first column dependence relation occurs at Y^k $(1 < k \leq n)$.

Proposition 3.5. *Assume $\mathcal{M}(n)(\beta)$ is positive, recursively generated, and $YX = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$. Suppose that for some k $(1 < k \leq n)$, $\mathcal{S} \equiv \{1, X, Y, ..., X^{k-1}, Y^{k-1}, X^k\}$ is linearly independent and $Y^k \in \operatorname{lin.span} \mathcal{S}$. Then $\mathcal{M}(n)$ admits a flat extension (and $\beta^{(2n)}$ admits a $\operatorname{rank}\mathcal{M}(n)$-atomic representing measure).*

Proof. In $\mathcal{C}_{\mathcal{M}(n)}$ we have a dependence relation

$$Y^k = a_0 1 + a_1 X + b_1 Y + ... + a_{k-1} X^{k-1} + b_{k-1} Y^{k-1} + a_k X^k. \qquad (3.13)$$

If $a_k = 0$, we can interchange the roles of X and Y and invoke Proposition 3.4. We may thus assume $a_k \neq 0$, so there is a dependence relation of the form

$$X^k = \tilde{a}_0 1 + \tilde{a}_1 X + \tilde{b}_1 Y + ... + \tilde{a}_{k-1} X^{k-1} + \tilde{b}_{k-1} Y^{k-1} + \tilde{b}_k Y^k. \qquad (3.14)$$

Since any flat extension $\mathcal{M}(n+1)$ must be recursively generated, with $YX = 0$, in $B(n+1)$ we must have

$$Y^{n+1} = a_0 Y^{n+1-k} + b_1 Y^{n+2-k} + ... + b_{k-1} Y^n \qquad (3.15)$$

and

$$X^{n+1} = \tilde{a}_0 X^{n+1-k} + \tilde{a}_1 X^{n+2-k} + ... + \tilde{a}_{k-1} X^n; \qquad (3.16)$$

let **w** denote the column vector of length $m(n)$ such that

$$\langle \mathbf{w}, Y^i X^j \rangle := \begin{cases} \tilde{a}_{j-n-1+k} & \text{if } n+1-k \leq j \leq n \text{ and } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

To show that Y^{n+1} is consistent with moment matrix structure and $YX = 0$, we first verify that

$$\langle Y^{n+1}, Y^i X^j \rangle = \beta_{n+1+i,j} \ (0 \le i + j \le n - 1); \tag{3.17}$$

indeed,

$$\begin{aligned}
\langle Y^{n+1}, Y^i X^j \rangle &= \langle a_0 Y^{n+1-k}, Y^i X^j \rangle + \langle b_1 Y^{n+2-k}, Y^i X^j \rangle + ... + \langle b_{k-1} Y^n, Y^i X^j \rangle \\
&= \langle a_0 Y^{n-k}, Y^{i+1} X^j \rangle + \langle b_1 Y^{n+1-k}, Y^{i+1} X^j \rangle + ... \\
&\quad + \langle b_{k-1} Y^{n-1}, Y^{i+1} X^j \rangle \\
&= \langle Y^n, Y^{i+1} X^j \rangle \ \text{ (by recursiveness, using (3.13))} \\
&= \beta_{n+i+1,j} \ \text{ (by the structure of } \mathcal{M}(n)\text{)}.
\end{aligned}$$

Since (3.15) readily implies that $\langle Y^{n+1}, Y^i X^j \rangle = 0$ when $j > 0$ and $i+j = n$, it follows that Y^{n+1} is consistent. A similar argument (using (3.14) and (3.16)) shows that X^{n+1} is also consistent with moment matrix structure and $YX = 0$. Now, setting $Y^i X^j = 0$ in $B(n+1)$ for $i+j = n+1$ $(i, j > 0)$, we have a moment matrix block $B(n+1)$ consistent with $YX = 0$ and satisfying Ran $B(n+1) \subseteq$ Ran $\mathcal{M}(n)$. Consider $\mathcal{M}\hat{} := [\mathcal{M}(n); B(n+1)] \equiv \begin{pmatrix} \mathcal{M}(n) & B(n+1) \\ B(n+1)^t & C \end{pmatrix}$. To show that $\mathcal{M}\hat{}$ is a moment matrix, it suffices to check that in block C, $C_{n+2,1} \equiv \langle X^{n+1}, Y^{n+1} \rangle = 0$, and this follows immediately from the identity $C_{n+2,1} = [Y^{n+1}]_{m(n)}^t \cdot \mathbf{w}$. $\qquad \square$

The following result concludes the proof of Theorem 3.1.

Proposition 3.6. *Assume that $\mathcal{M} \equiv \mathcal{M}(n)$ is positive, recursively generated, satisfies $YX = 0$, and that $\mathcal{S}_n(n)$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$. Then either $\mathcal{M}(n)$ admits a flat extension (and β admits a $(2n+1)$-atomic representing measure) or $\mathcal{M}(n)$ admits a rank-$(2n+2)$ positive, recursively generated extension $\mathcal{M}(n+1)$ which has a flat extension $\mathcal{M}(n+2)$ (and β admits a $(2n+2)$-atomic representing measure).*

Proof. By hypothesis, $\mathcal{B} := \{1, X, Y, ..., X^n, Y^n\}$ is a basis for $\mathcal{S}_n(n)$, so the compression $A \equiv [\mathcal{M}]_{\mathcal{B}}$ of $\mathcal{M}$ to the rows and columns of $\mathcal{B}$ is positive and invertible. We define the columns X^{n+1} and Y^{n+1} in the proposed block $B(n+1)$ by

$$\langle X^{n+1}, X^i \rangle := \begin{cases} \langle X^n, X^{i+1} \rangle \ (= \beta_{0,n+i+1}) & 0 \le i < n \\ p & i = n \end{cases}$$

$$\langle Y^{n+1}, Y^j \rangle := \begin{cases} \langle Y^n, Y^{j+1} \rangle \ (= \beta_{n+j+1,0}) & 0 \le j < n \\ q & j = n \end{cases}$$

and $\langle X^{n+1}, Y^j X^i \rangle := 0$ $(1 \le i + j \le n; j \ge 1)$ and $\langle Y^{n+1}, Y^j X^i \rangle := 0$ $(1 \le i + j \le n; i \ge 1)$, where p and q are two parameters. Let $\mathbf{r}_p := [X^{n+1}]_{\mathcal{B}}$ and $\mathbf{s}_q := [Y^{n+1}]_{\mathcal{B}}$. Due to the $YX = 0$ structure of $\mathcal{M}(n)$, it is straightforward to check that Ran $B(n + 1) \subseteq$ Ran $\mathcal{M}$, and that if W is a matrix satisfying

$\mathcal{M}W = B(n+1)$, then

$$C \equiv W^*\mathcal{M}W = \begin{pmatrix} \mathbf{r}_p^t A^{-1}\mathbf{r}_p & 0 & \cdots & 0 & \mathbf{r}_p^t A^{-1}\mathbf{s}_q \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \mathbf{s}_q^t A^{-1}\mathbf{r}_p & 0 & \cdots & 0 & \mathbf{s}_q^t A^{-1}\mathbf{s}_q \end{pmatrix}.$$

It follows at once that $\mathcal{M}$ admits a flat extension $\mathcal{M}(n+1)$ if and only if there exist real numbers p and q such that

$$\alpha(p, q) := \mathbf{s}_q^t A^{-1}\mathbf{r}_p = 0.$$

We may thus assume that α is nonzero on $\mathbb{R}^2$. Fix $p, q \in \mathbb{R}$ and let $u > \mathbf{r}_p^t A^{-1}\mathbf{r}_p$ and $v := \frac{\alpha^2}{u - \mathbf{r}_p^t A^{-1}\mathbf{r}_p} + \mathbf{s}_q^t A^{-1}\mathbf{s}_q$, so that

$$C(u, v) := \begin{pmatrix} u & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & v \end{pmatrix} \geq C(n+1)$$

and

$$\mathrm{rank}(C(u, v) - C) = 1.$$

Then

$$\mathcal{M}(n+1) \equiv \mathcal{M}(n+1; u, p, q) := \begin{pmatrix} \mathcal{M} & B(n+1) \\ B(n+1)^* & C(u, v) \end{pmatrix}$$

is positive and recursively generated, and $\mathrm{rank}\,\mathcal{M}(n+1) = 1 + \mathrm{rank}\,\mathcal{M} \ (= 2n+2)$.

We claim that $\mathcal{M}(n+1)$ admits a flat extension $\mathcal{M}(n+2)$. We first show that there is a unique block $B(n+2)$, subordinate to $YX = 0$, such that $\mathrm{Ran}\, B(n+2) \subseteq \mathrm{Ran}\,\mathcal{M}(n+1)$. In any such block, $YX^{n+1} = Y^2 X^n = \ldots = Y^{n+1}X = 0$. In Y^{n+2}, all the entries are determined from $\mathcal{M}(n+1)$ and $YX = 0$, except $s := \langle Y^{n+2}, Y^{n+1}\rangle$. Now, in $\mathcal{C}_{\mathcal{M}(n+1)}$ we have a dependence relation

$$Y^{n+1} = p_{n+1}(X) + q_n(Y),$$

with $\deg p_{n+1} \leq n+1, \deg q_n \leq n$. Since a flat extension must necessarily be recursively generated, in $\mathcal{C}_{\mathcal{M}(n+2)}$ we must have

$$Y^{n+2} = p_{n+1}(0)Y + Yq_n(Y), \tag{3.18}$$

whence $s = \langle p_{n+1}(0)Y + Yq_n(Y), Y^{n+1}\rangle$. Thus, $Y^{n+2} \in \mathrm{Ran}\,\mathcal{M}(n+1)$ and Y^{n+2} has $YX = 0$ structure and is Hankel with respect to $Y^{n+1}X$.

Note that if X^{n+2} for block $B(n+2)$ is defined to be consistent with known moment values and $YX = 0$ structure, then every value in X^{n+2} is determined except $r := \langle X^{n+2}, X^{n+1}\rangle$. We next show that there is a unique value of r such that $X^{n+2} \in \mathrm{Ran}\,\mathcal{M}(n+1)$. Since $u > \mathbf{r}_p^t A^{-1}\mathbf{r}_p^t$, Smul'jan's Theorem [Smu] (described in Section 1) implies that the compression $\tilde{A}$ of $\mathcal{M}(n+1)$ to rows and

columns given by $\tilde{\mathcal{B}} := \mathcal{B} \bigcup \{X^{n+1}\}$ is positive and invertible, and of the form $\tilde{A} = \begin{pmatrix} A & \mathbf{r}_p \\ \mathbf{r}_p^t & u \end{pmatrix}$. We now apply Lemma 3.3 to obtain

$$\tilde{A}^{-1} = \frac{1}{\delta} \begin{pmatrix} (\delta + A^{-1}\mathbf{r}_p\mathbf{r}_p^t)A^{-1} & -A^{-1}\mathbf{r}_p \\ -\mathbf{r}_p^t A^{-1} & 1 \end{pmatrix} \quad (\delta := u - \mathbf{r}_p^t A^{-1}\mathbf{r}_p).$$

Observe that $\left[X^{n+2}\right]_{\tilde{\mathcal{B}}} \equiv \begin{pmatrix} \left[X^{n+2}\right]_{\mathcal{B}} \\ r \end{pmatrix}$, and that, apart from the entries in $\left[X^{n+2}\right]_{\tilde{\mathcal{B}}}$, all other entries of X^{n+2} in $B(n+2)$ are zero. From the $YX = 0$ structure of $\mathcal{M}(n)$, it follows that $X^{n+2} \in \operatorname{Ran} \mathcal{M}(n+1)$ if and only if

$$\left\langle X^{n+2}, Y^{n+1} \right\rangle := \left[Y^{n+1}\right]_{\tilde{\mathcal{B}}}^t \tilde{A}^{-1} \left[X^{n+2}\right]_{\tilde{\mathcal{B}}} = 0, \tag{3.19}$$

so it suffices to show that (3.19) admits a unique solution for r. Now,

$$\begin{aligned}
\left[Y^{n+1}\right]_{\tilde{\mathcal{B}}}^t \tilde{A}^{-1} \left[X^{n+2}\right]_{\tilde{\mathcal{B}}} &= \begin{pmatrix} \mathbf{s}_q^t & 0 \end{pmatrix} \begin{pmatrix} A & \mathbf{r}_p \\ \mathbf{r}_p^t & u \end{pmatrix}^{-1} \begin{pmatrix} \left[X^{n+2}\right]_{\mathcal{B}} \\ r \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{s}_q^t & 0 \end{pmatrix} \frac{1}{\delta} \begin{pmatrix} (\delta + A^{-1}\mathbf{r}_p\mathbf{r}_p^t)A^{-1} & -A^{-1}\mathbf{r}_p \\ -\mathbf{r}_p^t A^{-1} & 1 \end{pmatrix} \begin{pmatrix} \left[X^{n+2}\right]_{\mathcal{B}} \\ r \end{pmatrix} \\
&= \frac{1}{\delta} \begin{pmatrix} \mathbf{s}_q^t & 0 \end{pmatrix} \begin{pmatrix} H - rA^{-1}\mathbf{r}_p \\ * \end{pmatrix} \quad \text{(for a certain vector } H\text{)} \\
&= \frac{1}{\delta}(\mathbf{s}_q^t H - r\mathbf{s}_q^t A^{-1}\mathbf{r}_p) = \frac{1}{\delta}(\mathbf{s}_q^t H - r\alpha(p,q)).
\end{aligned}$$

Since $\alpha(p,q) \neq 0$, it follows that (3.19) admits a unique solution $r \equiv r(p,q,u)$.

With this value, $\operatorname{Ran} B(n+1) \subseteq \operatorname{Ran} \mathcal{M}(n+1)$, so $B(n+2) = \mathcal{M}(n+1)W$ for some matrix W. To show that the flat extension $\mathcal{M}_{n+2} := [\mathcal{M}(n+1); B(n+2)] \equiv \begin{pmatrix} \mathcal{M}(n+1) & B(n+2) \\ B(n+2)^t & C_{n+2} \end{pmatrix}$ is a moment matrix, it now suffices to show that $\left\langle Y^{n+2}, X^{n+2} \right\rangle = 0$; this is because, by positivity of $\mathcal{M}_{n+2}$, $\left\langle X^{n+2}, Y^{n+2} \right\rangle = \left\langle Y^{n+2}, X^{n+2} \right\rangle$ and, by flatness, all other entries of C_{n+2} (except $\left\langle X^{n+2}, X^{n+2} \right\rangle$ and $\left\langle Y^{n+2}, Y^{n+2} \right\rangle$) are clearly zero. Now recall that $Y^{n+2} = p_{n+1}(0)Y + Yq_n(Y)$ in $\mathcal{C}_{(\mathcal{M}(n+1)\ B(n+2))}$ (by (3.18)), so by flatness, the same relation must hold in $\mathcal{C}_{\mathcal{M}_{n+2}}$. Thus,

$$\left\langle Y^{n+2}, X^{n+2} \right\rangle = \left\langle p_{n+1}(0)Y + Yq_n(Y), X^{n+2} \right\rangle = \left\langle X^{n+2}, p_{n+1}(0)Y + Yq_n(Y) \right\rangle = 0,$$

since, by the construction of X^{n+2} in $B(n+2)$ (consistent with the relation $YX = 0$), $\left\langle X^{n+2}, Y^{j+1} \right\rangle = 0$ $(0 \leq j \leq n)$. $\qquad\qquad\square$

Remark 3.7. Example 5.4 (below) illustrates a case of Proposition 3.6 where $\mathcal{M}(2)$ admits no flat extension, so the minimal representing measure is $(2n+2)$-atomic.

4. Proof of Theorem 1.5

We now turn to the proof of Theorem 1.5, which we restate for the sake of convenience. As in previous sections, it suffices to consider the cases $yx = 1$ and $yx = 0$.

Theorem 4.1. *Let $\beta \equiv \beta^{(2n)} : \beta_{00}, \beta_{01}, \beta_{10}, ..., \beta_{0,2n}, ..., \beta_{2n,0}$ be a family of real numbers, $\beta_{00} > 0$, and let $\mathcal{M}(n)$ be the associated moment matrix. Assume that $\mathcal{M}(n)$ is positive, recursively generated, and satisfies $YX = 1$ (resp. $YX = 0$). Then $\operatorname{rank} \mathcal{M}(n) \leq 2n + 1$, and the following statements are equivalent.*

  (i) *β admits a representing measure (necessarily supported in $yx = 1$, resp. $yx = 0$).*
  (ii) *β admits a representing measure with convergent moments up to degree $2n + 2$ (necessarily supported in $yx = 1$, resp. $yx = 0$).*
  (iii) *β admits a representing measure μ (necessarily supported in $yx = 1$, resp. $yx = 0$) such that $\operatorname{card} \operatorname{supp} \mu \leq 1 + \operatorname{rank} M(n)$. If $\operatorname{rank} M(n) \leq 2n$, then μ can be taken so that $\operatorname{card} \operatorname{supp} \mu = \operatorname{rank} M(n)$.*
  (iv) *$M(n)$ admits a positive, recursively generated extension $M(n + 1)$.*
  (v) *$M(n)$ admits a positive, recursively generated extension $M(n + 1)$, with $\operatorname{rank} M(n + 1) \leq 1 + \operatorname{rank} M(n)$, and $M(n + 1)$ admits a flat extension $M(n+2)$. If $\operatorname{rank} M(n) \leq 2n$, then $M(n)$ admits a flat extension $M(n+1)$.*
  (vi) *$\operatorname{rank} M(n) \leq \operatorname{card} V(\beta)$.*

To establish Theorem 4.1 we require the following result, whose proof is an adaptation of the proof of Proposition 3.4.

Proposition 4.2. *Suppose $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ is positive, $YX = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, and $\mathcal{M}(n)$ has a positive, recursively generated extension $\mathcal{M}(n+1)$. Suppose also that there exists k, $1 < k \leq n$, such that $\mathcal{S}_n(k - 1)$ is linearly independent and $X^k \in \operatorname{lin.span} \mathcal{S}_n(k - 1)$. Then $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n + 1)$.*

Proof. By hypothesis, we may write

$$X^k = a_0 1 + a_1 X + b_1 Y + ... + a_{k-1}X^{k-1} + b_{k-1}Y^{k-1} \quad (a_i, b_i \in \mathbb{R}). \qquad (4.1)$$

Assume first that not all coefficients b_j are zero, and let $m \leq k - 1$ be the largest integer such that $b_m \neq 0$. In $\mathcal{M}(n + 1)$ we can formally multiply (4.1) by Y to obtain

$$0 = YX^k = a_0 Y + b_1 Y^2 + ... + b_m Y^{m+1},$$

from which it follows that Y^{m+1} is a linear combination of columns associated to powers of y of lower degree, and, a fortiori, that the same is true of Y^n. If we instead formally multiply (4.1) by X, we see that X^n is a linear combination of columns associated to powers of x of lower degree. Thus, $\mathcal{M}(n)$ is flat. Since $\mathcal{M}(n+1)$ is a recursively generated extension of $\mathcal{M}(n)$, it must be a flat extension of $\mathcal{M}(n)$, and the result follows in this case.

We can thus assume that all coefficients b_j are zero, that is,

$$X^k = a_0 1 + a_1 X + ... + a_{k-1}X^{k-1}. \qquad (4.2)$$

If $a_0 \neq 0$, in $\mathcal{M}(n+1)$ we can formally multiply (4.2) by Y to obtain $Y = 0$, a contradiction. Thus, (4.2) does not involve the column 1, just as in (3.3). We may now continue exactly as in the proof of Proposition 3.4, since the part of that proof following (3.3) does not entail the variety condition rank $\mathcal{M}(n) \leq \text{card } \mathcal{V}(\beta)$. $\qquad \square$

Proof of Theorem 4.1. By [CuFi2, Theorem 2.1] and the equivalence of the moment problems for $M(n)(\gamma)$ and $\mathcal{M}(n)(\beta)$ [CuFi6, Proposition 1.12], we can assume that the columns 1, X and Y are linearly independent. Observe first that $(iii) \Rightarrow (ii) \Rightarrow (i)$ trivially, that $(i) \Rightarrow (vi)$ by [CuFi3, (1.7)], and that $(iii) \Leftrightarrow (v)$ by (1.8) and (1.5). Also, $(vi) \Rightarrow (iii)$ by Theorem 1.1, so (i), (ii), (iii), (v) and (vi) are equivalent. Since $(v) \Rightarrow (iv)$ is trivial, to complete the proof it suffices to establish $(iv) \Rightarrow (v)$.

Assume first that $\mathcal{M}(n)$ satisfies $YX = 1$. The cases when rank $\mathcal{M}(n) \leq 2n$ correspond to the column dependence relations that we considered in Cases I-III in the proof of Theorem 2.1, so we reconsider these dependence relations. First recall our hypotheses: $\mathcal{M}(n)$ is positive, recursively generated, $YX = 1$, and $\mathcal{M}(n)$ admits a positive recursively generated extension $\mathcal{M}(n+1)$. We need to show that $\mathcal{M}(n)$ admits a flat extension.

Case I. We have $\mathcal{S}_n(k-1)$ linearly independent and $X^k = p_{k-1}(X) + q_{k-1}(Y)$ in $\mathcal{C}_{\mathcal{M}(n)}$, with deg p_{k-1}, deg $q_{k-1} \leq k-1$. By the Extension Principle [Fia1], the same relation must hold in the column space of the positive extension $\mathcal{M}(n+1)$. Since $\mathcal{M}(n+1)$ is recursively generated, we must also have

$$X^{k-1} \equiv YX^k = Yp_{k-1}(X) + Yq_{k-1}(Y). \tag{4.3}$$

Let b_{k-1} be the coefficient of y^{k-1} in q_{k-1}. If $b_{k-1} = 0$, then (4.3) implies that $\mathcal{S}_n(k-1)$ is linearly dependent, a contradiction. Thus, $b_{k-1} \neq 0$, so (4.3) implies that Y^k can be written as a linear combination of previous columns, and therefore $\mathcal{M}(k)$ is flat; it now follows from [Fia1] and recursiveness that $\mathcal{M}(n)$ is flat, and thus admits a flat extension.

Case II. Suppose $\mathcal{S} := \mathcal{S}_n(k-1) \bigcup \{X^k\}$ is linearly independent and $Y^k \in$ lin.span $\mathcal{S}$, for some $k < n$. The hypothesis about $\mathcal{M}(n+1)$ is superfluous, as we showed in Proposition 2.6 that $\mathcal{M}(n)$ is flat.

Case III. Here $\mathcal{S}_n(n-1) \bigcup \{X^n\}$ is linearly independent and

$$Y^n = a_n X^n + p_{n-1}(X) + q_{n-1}(Y) \tag{4.4}$$

in $\mathcal{C}_{\mathcal{M}(n)}$, with deg p_{n-1}, deg $q_{n-1} \leq n-1$. The Extension Principle [Fia1] shows that the same relation must hold in the column space of the positive extension $\mathcal{M}(n+1)$. Since $\mathcal{M}(n+1)$ is recursively generated, we must also have

$$Y^{n-1} \equiv Y^n X = a_n X^{n+1} + p_{n-1}(X)X + q_{n-1}(Y)X. \tag{4.5}$$

If $a_n = 0$, $\mathcal{S}_n(n-1) \bigcup \{X^n\}$ is linearly dependent, a contradiction. Thus, $a_n \neq 0$, which implies that X^{n+1} can be written in terms of columns of lower degree. Moreover, from (4.4) we obtain $Y^{n+1} = a_n X^{n-1} + Yp_{n-1}(X) + Yq_{n-1}(Y)$, so Y^{n+1} is also a linear combination of columns of lower degree. Finally, from $YX = 1$ and the recursiveness of $\mathcal{M}(n+1)$, we see that the intermediate columns $Y^i X^j$ ($i+j =$

$n + 1$, with $i, j \geq 1$) are all identical to columns corresponding to monomials of degree $n - 1$. It follows that $\mathcal{M}(n+1)$ is flat, and is thus a flat extension of $\mathcal{M}(n)$.

Case IV. Here $\mathcal{S}_{n-1}(n) \bigcup \{X^n\}$ we consider the case when $\mathcal{S}_n(n)$ is linearly independent; the result follows directly from Proposition 2.14 (without using the given extension $\mathcal{M}(n + 1)$). This completes the proof for $YX = 1$.

We now assume that $\mathcal{M}(n)$ is positive, recursively generated, $YX = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, and $\mathcal{M}(n)$ admits a positive, recursively generated extension $\mathcal{M}(n+1)$. We consider again the various cases of column dependence relations that we examined in the proof of Theorem 3.1. If the first dependence relation in $\mathcal{S}_n(n)$ occurs at X^k ($2 \leq k \leq n$), then Proposition 4.2 implies that $\mathcal{M}(n)$ has a flat extension $\mathcal{M}(n + 1)$. If the first dependence relation occurs at Y^k ($2 \leq k \leq n$), then, without recourse to the given extension $\mathcal{M}(n + 1)$, Proposition 3.5 implies that $\mathcal{M}(n)$ admits a flat extension. In the remaining case, $\text{rank}\,\mathcal{M}(n) = 2n + 1$, so the result follows from Proposition 3.6 (again, without using the given extension $\mathcal{M}(n + 1)$). □

5. Some Examples Illustrating Theorems 2.1 and 3.1

Example 1.6 illustrates Case III of Theorem 2.1. We now present examples corresponding to other cases of Theorems 2.1 and 3.1.

Example 5.1. (Theorem 2.1, Case I) We illustrate $\mathcal{M}(3)$ in which $YX = 1$ and the first dependence relation in $\mathcal{S}_3(3)$ occurs at X^3. We define

$$\mathcal{M}(3) \equiv \mathcal{M}(3)(\beta) := \begin{pmatrix} 1 & 0 & 0 & a & 1 & a & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 & 2a^2 & a & 1 & a \\ 0 & 1 & a & 0 & 0 & 0 & a & 1 & a & 2a^2 \\ a & 0 & 0 & 2a^2 & a & 1 & e & 0 & 0 & 0 \\ 1 & 0 & 0 & a & 1 & a & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & a & 2a^2 & 0 & 0 & 0 & f \\ 0 & 2a^2 & a & e & 0 & 0 & g & 2a^2 & a & 1 \\ 0 & a & 1 & 0 & 0 & 0 & 2a^2 & a & 1 & a \\ 0 & 1 & a & 0 & 0 & 0 & a & 1 & a & 2a^2 \\ 0 & a & 2a^2 & 0 & 0 & f & a & a & 2a^2 & h \end{pmatrix},$$

where $a > 1$. Clearly $YX = 1$, and a calculation shows that $\text{rank}\,\mathcal{M}(2) = 5$. Thus, the set $\mathcal{B} := \{1, X, Y, X^2, Y^2\}$ is linearly independent, and

$$[X^3]_{\mathcal{B}} = -ae[1]_{\mathcal{B}} + \frac{a(2a^2 - 1)}{a^2 - 1}[X]_{\mathcal{B}} - \frac{a^2}{a^2 - 1}[Y]_{\mathcal{B}} + \frac{a^2 e}{2a^2 - 1}[X^2]_{\mathcal{B}} + \frac{e(a^2 - 1)}{2a^2 - 1}[Y^2]_{\mathcal{B}}. \tag{5.1}$$

We seek to impose the same relation in $\mathcal{C}_{\mathcal{M}(3)}$, which entails

$$g = \frac{a(2a^2 - 1)}{a^2 - 1}2a^2 - \frac{a^2}{a^2 - 1}a + \frac{a^2 e}{2a^2 - 1}e = a^2 \frac{8a^5 - 10a^3 + 3a + a^2 e^2 - e^2}{(a^2 - 1)(2a^2 - 1)}$$

and

$$\frac{a(2a^2-1)}{a^2-1}a - \frac{a^2}{a^2-1}2a^2 + \frac{e(a^2-1)}{2a^2-1}f = a,$$

so that

$$f = a\frac{(a^2+a-1)(2a^2-1)}{e\,(a^2-1)^2}.$$

To compute $\mathcal{V}(\mathcal{M}(3))$, note that the column relation (5.1), together with $YX = 1$, gives rise to the equation

$$x^3 + ae - \frac{a(2a^2-1)}{a^2-1}x + \frac{a^2}{a^2-1}\frac{1}{x} - \frac{a^2e}{2a^2-1}x^2 - \frac{e(a^2-1)}{2a^2-1}\frac{1}{x^2} = 0,$$

or, equivalently,

$$(2a^2-1)(a^2-1)x^5 - a^2e(a^2-1)x^4 - a(2a^2-1)^2x^3 \qquad (5.2)$$
$$+ae(a^2-1)(2a^2-1)x^2 + a^2(2a^2-1)x - e(a^2-1)^2 = 0.$$

Thus, $\mathcal{V}(\mathcal{M}(3))$ can have at most five points, so if rank $\mathcal{M}(3) \leq \operatorname{card} \mathcal{V}(\mathcal{M}(3))$, then column Y^3 must be written in terms of the columns in $\mathcal{B}$ (in particular, h is fully determined), and $\mathcal{M}(3)$ is therefore flat, in accordance with Proposition 2.5. For a specific numerical example, let $a := 2$ and $e := 1$, so that $f = \frac{70}{9}$, $g = \frac{740}{21}$, $h = \frac{2190400}{3087}$ and $Y^3 = -\frac{1480}{21}\cdot 1 - \frac{4}{3}X + \frac{14}{3}Y + \frac{740}{49}X^2 + \frac{2960}{147}Y^2$. Then (5.2) becomes $21x^5 - 12x^4 - 98x^3 + 42x^2 + 28x - 9 = 0$, which has five real roots, as follows: $x_1 \cong -2.0292$, $x_2 \cong -0.521229$, $x_3 \cong 0.282006$, $x_4 \cong 0.658788$ and $x_5 \cong 2.18106$. To calculate the densities we use Theorem 1.7; here V is the 5×5 matrix whose entry in row k, column ℓ is $y_\ell^{i_k}x_\ell^{j_k}$ ($1 \leq k, \ell \leq r$), where $(i_1, j_1) = (0,0)$, $(i_2, j_2) = (0,1)$, $(i_3, j_3) = (1,0)$, $(i_4, j_4) = (0,2)$ and $(i_5, j_5) = (2,0)$. We then solve the equation $V\rho^t = (\beta_{i_1,j_1}, ..., \beta_{i_5,j_5})^t$, where $\rho \equiv (\rho_1, ..., \rho_5)$ and $(\beta_{i_1,j_1}, ..., \beta_{i_5,j_5}) = (1,0,0,2,2)$. Thus, $\rho_1 \cong 0.228429$, $\rho_2 \cong 0.263185$, $\rho_3 \cong 0.0174322$, $\rho_4 \cong 0.31204$, $\rho_5 \cong 0.178914$. $\qquad\square$

Example 5.2. (Theorem 2.1, Cases III and IV) Consider the moment matrix

$$\mathcal{M}(3) \equiv \mathcal{M}(3)(\beta) := \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 3 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 2 & 5 \\ 1 & 0 & 0 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 3 \\ 0 & 3 & 1 & 0 & 0 & 0 & 14 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 3 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 2 & 5 \\ 0 & 2 & 5 & 0 & 0 & 3 & 1 & 2 & 5 & 33 \end{pmatrix}.$$

It is easy to see that $\mathcal{M}(3) \geq 0$, $YX = 1$ in $\mathcal{C}_{\mathcal{M}(3)}$, rank $\mathcal{M}(3) = 7$, and that $\mathcal{S}_3(3)$ is a basis for $\mathcal{C}_{\mathcal{M}(3)}$. The block $B(4)$ for a recursively generated extension is determined by the choice of $\beta_{07} = t$ and $\beta_{70} = u$. Following the proof of Proposition 2.14, we have Ran $B(4) \subseteq$ Ran $\mathcal{M}(3)$, and a calculation of $\mathcal{M} \equiv$

$[\mathcal{M}(3); B(4)]$ shows that $\mathcal{M}$ has the form of a moment matrix $\mathcal{M}(4)$ if and only if $(t+3)(u-150) = 1$. For example, with $t = -2$ and $u = 151$, we find the following column relations in $\mathcal{M}(4)$ that are not determined recursively from relations in $\mathcal{M}(3)$:

$$X^4 = -20 \cdot 1 + 4X + Y + 9X^2 + 7Y^2 - X^3 - Y^3$$

and

$$Y^4 = 4 \cdot 1 + 9X - 20Y - X^2 + Y^2 - X^3 + 7Y^3.$$

Together with $YX = 1$, these relations show that $\operatorname{card} \mathcal{V}(\mathcal{M}(4)) = \operatorname{rank} \mathcal{M}(4) = 7$; the x-coordinates of the points in $\{(x_i, \frac{1}{x_i})\}_{i=1}^{7} \equiv \mathcal{V}(\mathcal{M}(4))$ are as follows: $x_1 \cong -2.82238$, $x_2 \cong -1.87650$, $x_3 \cong -0.64947$, $x_4 \cong 0.14873$, $x_5 \cong 0.66898$, $x_6 \cong 1.32445$ and $x_7 \cong 2.20619$. Since $\mathcal{S}_3(3)$ is a basis for $\mathcal{C}_{\mathcal{M}(3)}$, Theorem 1.7 now implies that the densities $\{\rho_i\}_{i=1}^{7}$ of the unique representing measure $\mu = \sum_{i=1}^{7} \rho_i \delta_{(x_i, \frac{1}{x_i})}$ for $\mathcal{M}(4)$ are as follows: $\rho_1 \cong 0.00754$, $\rho_2 \cong 0.07579$, $\rho_3 \cong 0.41491$, $\rho_4 \cong 0.00025$, $\rho_5 \cong 0.43154$, $\rho_6 \cong 0.01146$ and $\rho_7 \cong 0.05851$.

We can illustrate the second part of Case IV and also Case III if we use $t = -3$ and $u = 150$, so that M is not a moment matrix. In this case, the C block of M satisfies $C_{11} = 79$. Following the proof of Proposition 2.14, we redefine $C_{11} := 80$, $C_{15} := 1$, and we compute C_{55} ($= \beta_{80}$) $= 1036$, so as to define a positive rank 8 moment matrix $M(4)$ in which the first dependence relation in $\mathcal{S}_4(4)$ is of the form $Y^4 = -16 \cdot 1 + 18X - 21Y + 8X^2 + 8Y^2 - 3X^3 + 6Y^3 - X^4$ (i.e., $M(4)$ is in Case III). This relation and $YX = 1$ determine $\mathcal{V} \equiv \mathcal{V}(\mathcal{M}(4))$, and a calculation shows that $\operatorname{card} \mathcal{V} = \operatorname{rank} \mathcal{M}(4) = 8$; the x-coordinates of the points $\{(x_i, \frac{1}{x_i})\}_{i=1}^{8} \equiv \mathcal{V}$ are as follows: $x_1 \cong -3.63582$, $x_2 \cong -2.02578$, $x_3 \cong -0.840968$, $x_4 \cong -0.644996$, $x_5 \cong 0.149637$, $x_6 \cong 0.670287$, $x_7 \cong 1.14158$ and $x_8 \cong 2.18606$. Since the unique representing measure μ for the flat extension $M(5)$ of $M(4)$ (guaranteed by Proposition 2.7) satisfies $\operatorname{card} \operatorname{supp} \mu = \operatorname{rank} M(4) = 8$, it follows that $\operatorname{supp} \mu = \mathcal{V}$ and that the densities of μ (computed as in Theorem 1.7) are $\rho_1 \cong 0.00087$, $\rho_2 \cong 0.07462$, $\rho_3 \cong 0.02883$, $\rho_4 \cong 0.39396$, $\rho_5 \cong 0.00025$, $\rho_6 \cong 0.43357$, $\rho_7 \cong 0.00612$ and $\rho_8 \cong 0.06177$. $\qquad \square$

We next turn to examples illustrating Theorem 3.1.

Example 5.3. (cf. Proposition 3.4) We illustrate $\mathcal{M}(2)$ with $YX = 0$, where the first dependence relation in $\mathcal{S}_2(2)$ occurs at X^2. Consider the moment matrix

$$\mathcal{M}(2) := \begin{pmatrix} 1 & a & 0 & b & 0 & d \\ a & b & 0 & e & 0 & 0 \\ 0 & 0 & d & 0 & 0 & g \\ b & e & 0 & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & g & 0 & 0 & h \end{pmatrix}.$$

A straightforward calculation reveals that with $b > a^2$, $d > 0$, $e = \frac{b^2}{a}$, $f = \frac{b^3}{a^2}$ and $h > \frac{bd^3 - a^2g^2 + bg^2}{d(b-a^2)}$, $\mathcal{M}(2) \geq 0$, $\operatorname{rank} \mathcal{M}(2) = 4$, $YX = 0$ in $\mathcal{C}_{\mathcal{M}(2)}$, and $X^2 = \frac{b}{a}X$. The variety $\mathcal{V}(\mathcal{M}(2))$ associated to this matrix consists of the intersection of $yx = 0$

with $x(x - \frac{b}{a}) = 0$, so card $\mathcal{V}(\mathcal{M}(2)) = \infty$. Now, in any recursively generated extension $\mathcal{M}(3)$ we must have $X^3 = \frac{b}{a}X^2 = \frac{b^2}{a^2}X$. Also, the form of a block $B(3)$ is

$$B(3) \equiv \begin{pmatrix} e & 0 & 0 & g \\ f & 0 & 0 & 0 \\ 0 & 0 & 0 & h \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

where $p := \frac{b^4}{a^3}$ and q is free. Following the proof of Proposition 3.4, the existence of a flat extension $\mathcal{M}(3)$ depends on the verification of (3.9), that is, $[X^3]_{\{1,2,3,6\}} \mathcal{M}(2)^{-1}_{\{1,2,3,6\}} [Y^3]_{\{1,2,3,6\}} = 0$. A straightforward calculation shows that this is indeed the case, so $\mathcal{M}(2)$ admits a flat extension. $\qquad\square$

Example 5.4. (cf. Proposition 3.6 and [CuFi6, Example 5.6]) Consider the moment matrix

$$\mathcal{M}(2) := \begin{pmatrix} 1 & 1 & 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 4 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 9 \\ 2 & 4 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 9 & 0 & 0 & 28 \end{pmatrix}$$

Observe that rank $\mathcal{M}(2) = 5$, and that $YX = 0$ in $\mathcal{C}_{\mathcal{M}(2)}$, so that $\mathcal{S}_2(2)$ is a basis for $\mathcal{C}_{\mathcal{M}(2)}$. A block $B(3)$ for a recursively generated extension is completely determined by the choice of $\beta_{05} = p$ and $\beta_{50} = q$. With these choices, the C block in the flat extension $[\mathcal{M}(2); B(3)]$ has entries $C_{11} = (p - 18)^2 + 42$, $C_{14} = C_{41} = 1$, and $C_{44} = (q - 84)^2 + 262$. It is then clear that $[\mathcal{M}(2); B(3)]$ is not a moment matrix, so $\beta^{(4)}$ has no 5-atomic representing measure. To construct a 6-atomic representing measure, we modify the two key entries, C_{11} and C_{14}, as dictated by the proof of Proposition 3.6. By taking $p = 18$, $q = 84$, $u = 43$, it is easy to see that rank $\mathcal{M}(3) = 1 + $ rank $\mathcal{M}(2) \ (= 6)$ precisely when $v = 263$. As in the proof of Proposition 3.6, we claim that $\mathcal{M}(3)$ admits a flat extension $\mathcal{M}(4)$. We first exhibit a unique block $B(4)$, subordinate to $YX = 0$, such that Ran $B(4) \subseteq$ Ran $\mathcal{M}(3)$. In any such block, $YX^3 = Y^2X^2 = Y^3X = 0$. In Y^4, all the entries are determined from $\mathcal{M}(3)$ and $YX = 0$, except $s := \langle Y^4, Y^3 \rangle$. Now, in $\mathcal{C}_{\mathcal{M}(3)}$ we have a dependence relation

$$Y^3 = -5 \cdot 1 + 7X + 11Y - X^3;$$

since a flat extension must necessarily be recursively generated, in $\mathcal{C}_{\mathcal{M}(4)}$ we must have

$$Y^4 = -5Y + 11Y^2,$$

whence $s = \langle -5Y + 11Y^2, Y^3 \rangle = -5\beta_{4,0} + 11\beta_{5,0} = 784$. Also, in X^4, all values are determined except $r := \langle X^4, X^3 \rangle$; following Proposition 3.6, we claim that

there is a unique value for r such that $X^4 \in \text{Ran } \mathcal{M}(3)$. For, relative to the linearly independent set of columns $\{1, X, Y, X^2, Y^2, X^3\}$ in $\mathcal{C}_{\mathcal{M}(3)}$,

$$X^4 = (3r - 243) \cdot 1 + (481 - 6r)X + (81 - r)Y + 7X^2 + (r - 81)X^3,$$

and since $\langle X^4, Y^3 \rangle$ must be zero, we obtain

$$0 = \langle (3r - 243) \cdot 1 + (481 - 6r)X + (81 - r)Y + 7X^2 + (r - 81)X^3, Y^3 \rangle$$
$$= (3r - 243)\beta_{3,0} + (81 - r)\beta_{4,0} = (3r - 243) \cdot 9 + (81 - r) \cdot 28 = 81 - r,$$

whence $r = 81$. The unique 6-atomic representing measure associated with $\mathcal{M}(3)$ is then given by $(x_1, y_1) \cong (2.16601, 0)$, $(x_2, y_2) \cong (0.782816, 0)$, $(x_3, y_3) \cong (-2.94883, 0)$, $(x_4, y_4) \cong (0, 0.463604)$, $(x_5, y_5) \cong (0, 3.06043)$, $(x_6, y_6) \cong (0, -3.52404)$, $\rho_1 \cong 0.393081$, $\rho_2 \cong 0.203329$, $\rho_3 \cong 0.00359018$, $\rho_4 \cong 0.0821253$, $\rho_5 \cong 0.316218$, $\rho_6 \cong 0.00165656$. $\qquad\square$

6. An Application to the Full Moment Problem

We conclude with a new proof of Theorem 1.3. We require the following preliminary result, which was given in [Fia3] for complex moment matrices, but also holds for $\mathcal{M}(\infty)(\beta)$.

Lemma 6.1. *([Fia3, Proposition 4.2])   Let $\beta \equiv \beta^{(\infty)}$ be a full sequence such that $\mathcal{M}(\infty) \geq 0$. Then $\mathcal{M}(n)$ is positive and recursively generated for each $n \geq 1$.*

Theorem 6.2. *(cf. [Sto1])   Let $P \in \mathbb{R}[x, y]$ with $\deg P \leq 2$. The sequence $\beta \equiv \beta^{(\infty)}$ has a representing measure supported in $P(x, y) = 0$ if and only if $\mathcal{M} \equiv \mathcal{M}(\infty)(\beta) \geq 0$ and $P(X, Y) = 0$ in $\mathcal{C}_\mathcal{M}$.*

Proof. Let μ be a representing measure for β supported in $\mathcal{Z}(P)$. For each $n \geq 2$, μ is a representing measure for $\beta^{(2n)}$, so (1.4) and (1.5) imply that $\mathcal{M}(n)(\beta) \geq 0$ and $P(X, Y) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$; thus $\mathcal{M} \geq 0$ and $P(X, Y) = 0$ in $\mathcal{C}_\mathcal{M}$.

For the converse, since $\mathcal{M} \geq 0$, Lemma 6.1 implies that for $n \geq 2$, $\mathcal{M}(n+1)$ is positive and recursively generated, so $\mathcal{M}(n)$ has a positive, recursively generated extension. Since $P(X, Y) = 0$ in $\mathcal{C}_\mathcal{M}$, the same is true in $\mathcal{C}_{\mathcal{M}(n)}$. Suppose now that $P(x, y) = 0$ is a nondegenerate hyperbola. Let Φ be an injective degree-one mapping of the plane onto itself such that $\Phi(\mathcal{Z}(P)) = \{(x, y) : yx = 1\}$ (cf. Section 1), and let $\tilde{\beta}^{(2n)}$ be the sequence corresponding to $\beta^{(2n)}$ via Proposition 1.9. Proposition 1.9(ix) implies that $\tilde{\mathcal{M}}(n) \equiv \mathcal{M}(n)(\tilde{\beta})$ has a positive, recursively generated extension $\tilde{\mathcal{M}}(n + 1)$, and Proposition 1.9(vi) implies that $YX = 1$ in $\mathcal{C}_{\tilde{\mathcal{M}}(n)}$. Theorem 1.5(iv) now implies that $\tilde{\beta}^{(2n)}$ has a representing measure supported in $yx = 1$, so Proposition 1.9(v) implies that $\beta^{(2n)}$ has a representing measure supported in $P(x, y) = 0$; the existence of a representing measure for β supported in $P(x, y) = 0$ now follows from Theorem 1.4.

The cases of other conics are handled similarly. For degenerate hyperbolas, we use the case of Theorem 1.1 for $yx = 0$. For parabolas and ellipses we use

[CuFi7, Theorem 1.4] and [CuFi5, Theorem 3.5], respectively. The cases of lines is treated directly in [CuFi2, Theorem 2.1]. □

Remark 6.3. (i) By analogy with properties (A) and (A'_n) (cf. Section 1), consider also the following possible properties for a polynomial $P \in \mathbb{R}[x, y]$:

$\beta \equiv \beta^{(2n)}$ has a representing measure supported in $\mathcal{Z}(P)$ if and only if

$$\mathcal{M}(n)(\beta) \text{ is positive semi-definite, recursively generated} \qquad (A_n)$$
$$\text{and } P(X, Y) = 0 \text{ in } \mathcal{C}_{\mathcal{M}(n)}.$$

$\beta \equiv \beta^{(\infty)}$ has a representing measure supported in $\mathcal{Z}(P)$ if and only if

$$\mathcal{M}(\infty)(\beta) \text{ is positive semi-definite, } P(X, Y) = 0 \text{ in } \mathcal{C}_{\mathcal{M}(\infty)}, \qquad (A')$$
$$\text{and } \operatorname{rank} \mathcal{M}(\infty) \leq \operatorname{card} \mathcal{V}(\mathcal{M}(\infty)).$$

Theorem 1.4 readily implies the following implications concerning possible properties enjoyed by a given polynomial P:

$$(A_n) \text{ holds for all } n \geq \deg P \quad \Rightarrow \quad (A'_n) \text{ holds for all } n \geq \deg P$$
$$\Downarrow \qquad\qquad\qquad\qquad\qquad \Downarrow$$
$$(A) \text{ holds} \qquad\qquad \Rightarrow \qquad\qquad (A') \text{ holds.}$$

First-degree polynomials P satisfy (A_n) for all $n \geq \deg P$, but our results show that second-degree polynomials do not. However, second-degree polynomials satisfy (A'_n) for all $n \geq \deg P$, and also (A). Stochel [Sto1] has identified cubics which fail to satisfy (A), but we know of no example of a polynomial P that fails to satisfy (A'), or (A'_n) for some $n \geq \deg P$.

(ii) The full moment problem on compact semi-algebraic sets in $\mathbb{R}^n$ was solved by K. Schmüdgen [Schm1]. Recently, the analysis of the semi-algebraic case was extended to non-compact sets by V. Powers and C. Scheiderer [PoSc] (cf. [KuMa], [Sche], [Schm2]).

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