Standard Operator Models in the Polydisc, II

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We first show that every γ-contractive commuting multioperator is unitarily equivalent to the restriction of $S^{(\gamma)} \oplus W$ to an invariant subspace, where $S^{(\gamma)}$ is a backwards multi-shift and W a γ-isometry. We then describe γ-isometries in terms of $(\gamma, 1)$-isometries, and establish that under an additional assumption on T, W above can be chosen to be a commuting multioperator of isometries. Our methods provide, as a by-product, a new proof of the existence of a regular unitary dilation for every $(1, ..., 1)$-contractive commuting multioperator.

1. Introduction. The present paper continues and completes the work in [CuVa]; for the reader's convenience, we recall here the terminology and some basic facts.

Let $\mathcal{H}$ be a complex Hilbert space, let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$, and let $n \geq 1$ be a fixed integer. If $T = (T_1, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^n$ is a commuting multioperator (abbreviated c.m.), then for every $\gamma \in \mathbf{Z}_+^n$ we set

$$\Delta_T^\gamma := \sum_{\alpha \leq \gamma} (-1)^{|\alpha|} \frac{\gamma!}{\alpha!(\gamma - \alpha)!} T^{*\alpha} T^\alpha, \tag{1.1}$$

where, as usual, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\alpha! := \alpha_1! \cdots \alpha_n!$, $T^\alpha := T_1^{\alpha_1} \cdots T_n^{\alpha_n}$, and $T^* := (T_1^*, \ldots, T_n^*)$.

If we associate T with the commuting operators $M_{T_j} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ given by

$$M_{T_j}(X) := T_j^* X T_j \quad (X \in \mathcal{L}(\mathcal{H}),\ j = 1, \ldots, n), \tag{1.2}$$

then we have

$$\Delta_T^\gamma = (I - M_{T_1})^{\gamma_1} \cdots (I - M_{T_n})^{\gamma_n}(1), \tag{1.3}$$

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where 1 is the identity on $\mathcal{H}$, and I is the identity of $\mathcal{L}(\mathcal{H})$.

Let $e \equiv e^{(n)} := (1, \ldots, 1) \in \mathbf{Z}_+^n$ and let $\gamma \geq e^{(n)}$ be a multi-index, which will remain fixed throughout the paper. In [CuVa] we began to describe the structure of those c.m. $T \in \mathcal{L}(\mathcal{H})^n$ with the property $\Delta_T^\alpha \geq 0$ for all $\alpha \leq \gamma$, up to the so-called polydisc isometries (a polydisc isometry W is a c.m. consisting of contractions, and such that $\Delta_W^{e^{(n)}} = 0$). In the present work we shall give the definitive form of this structure result, including the description of the involved polydisc isometries.

We shall adopt the following terminology.

Definition 1.1 *Let $\gamma \geq e^{(n)}$ be given. A c.m. $T \in \mathcal{L}(\mathcal{H})^n$ is said to be γ-contractive if $\Delta_T^\alpha \geq 0$ for all $\alpha \leq \gamma$. A γ-contractive c.m. T is said to be a γ-isometry if $\Delta_T^{e^{(n)}} = 0$.*

If T is γ-contractive, then each T_i is a contraction $(i = 1, ..., n)$; for, if $e_j := (0, ..., \overset{j}{1}, ..., 0)$, then $e_j \leq \gamma$, and so $\Delta_T^{e_j} = 1 - T_j^* T_j \geq 0$ $(j = 1, \ldots, n)$.

It is also clear that a γ-isometry is a polydisc isometry, but the converse is not true in general. Indeed, if $T_1 \in \mathcal{L}(\mathcal{H})$ is a contraction such that $(I - M_{T_1})^2(1)$ is not positive (such operators do exist, see for instance [Agl]), and if $T_2 = 1$, then $T = (T_1, T_2)$ is a polydisc isometry which is not a $(2,1)$-isometry (since T is not $(2,1)$-contractive).

As in [CuVa], for a fixed $\gamma \geq e^{(n)}$ we consider the standard model $S^{(\gamma)}$ defined in the following way. Let $\mathcal{K} := \ell^2(\mathbf{Z}_+^n, \mathcal{H})$ (the Hilbert space consisting of those functions $f : \mathbf{Z}_+^n \to \mathcal{H}$ such that $\Sigma_{\alpha \in \mathbf{Z}_+^n} \parallel f(\alpha) \parallel^2 < \infty$), and let

$$(1.4) \qquad (S_j^{(\gamma)} f)(\alpha) \; := \; \left[\frac{\rho_\gamma(\alpha)}{\rho_\gamma(\alpha + e_j)} \right]^{1/2} f(\alpha + e_j) \qquad (\alpha \in \mathbf{Z}_+^n, \; j = 1, \ldots, n),$$

where

$$(1.5) \qquad \rho_\gamma(\alpha) \; := \; \frac{(\gamma + \alpha - e)!}{\alpha!(\gamma - e)!} \qquad (\alpha \in \mathbf{Z}_+^n).$$

Then $S^{(\gamma)} := (S_1^{(\gamma)}, \ldots, S_n^{(\gamma)})$ is a c.m. on $\mathcal{K}$, also called the backwards multi-shift of type (n, γ) [CuVa].

We can now state the following structure result (see also [CuVa, Theorem 3.15]).

Theorem 1.2 *Let $T \in \mathcal{L}(\mathcal{H})^n$ be a c.m. The following conditions are equivalent.*

(1) *T is γ–contractive.*

(2) *T is unitarily equivalent to the restriction of $S^{(\gamma)} \oplus W$ to an invariant subspace, where W is a γ–isometry.*

The proof of the implication (1)⇒(2) is a more refined version of the proof of the implication (a) ⇒ (b) from [CuVa, Theorem 3.15] (where it was only shown that W is a polydisc isometry). The proof of the implication (2) ⇒ (1) is relatively easy, via Lemma 2.3. (Theorem 3.15 in [CuVa] is slightly inaccurate at this point, since the proof passes through a third stronger condition.) We shall give a detailed proof of Theorem 1.2 in Section 2, along with the investigation of the structure of γ-isometries. In Section 3 we present some related results, together with a new proof of the existence of regular unitary dilations (see [SzFo, Theorem I.9.1]).

We note that if $n = 1$ and $\gamma = (m)$ $(m \geq 1)$, then $T = (T_1)$ is γ-contractive if and only if T_1 is an m-hypercontraction in the sense of [Agl], so Theorem 1.2 is an extension of [Agl, Theorem 1.10]. In particular, T_1 is a 1-hypercontraction if and only if T_1 is a contraction, and Theorem 1.2 also extends an assertion from [SzFo, I.10.1] which was, in fact, the starting point of our investigations. Finally, if $n \geq 1$ is arbitrary and $\gamma = (1, ..., 1) \in \mathbf{Z}_+^n$, then $T = (T_1, ..., T_n)$ γ-contractive means precisely that T satisfies Brehmer's condition for the existence of regular dilations for commuting contractions (see [Bre] or [SzFo]).

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2. Completing the structure theorem In this section we refine some auxiliary results from [CuVa] which are needed to obtain a specific decomposition of γ-isometries into simpler objects, up to unitary equivalence. The proof of Theorem 1.2 is a by-product of these arguments. The basic operator J_0, constructed in the next lemma, is a useful tool which will permit us to decompose every γ-contractive multioperator into a direct sum of two γ-contractive multioperators of a simpler form (see also Lemma 2.2).

Lemma 2.1 *Let $\gamma \geq e^{(n)}$, let $T \in \mathcal{L}(\mathcal{H})^n$ be γ-contractive, and let $p \in \{1, \ldots, n\}$. Then the limit*

$$J_0 h := \lim_{r \to 1^-} \prod_{j=1}^{p} (I - rM_{T_j})^{-\gamma_j} (I - M_{T_j})^{\gamma_j} (1) h \tag{2.1}$$

exists for every $h \in \mathcal{H}$, and we have $0 \leq J_0 \leq 1$. Moreover, for every $\beta \leq \gamma$,

$$\prod_{j=1}^{n} (I - M_{T_j})^{\beta_j} (J_0) \geq 0, \tag{2.2}$$

$$\prod_{j=1}^{n} (I - M_{T_j})^{\beta_j} (1 - J_0) \geq 0. \tag{2.3}$$

Proof. As in the proof of [CuVa, Corollary 3.7], we have

$$0 \le s\text{–}\lim_{r \to 1^-} \prod_{j=1}^{p} (I - rM_{T_j})^{-\gamma_j} (I - M_{T_j})^{\gamma_j}(1) \le 1,$$

showing that J_0 exists and $0 \le J_0 \le 1$ (s–lim denotes limit in the strong operator topology of $\mathcal{L}(\mathcal{H})$).

For a fixed $\beta \le \gamma$ we have

$$\begin{aligned} \text{(2.4)} \qquad & \prod_{j=1}^{n} (I - M_{T_j})^{\beta_j} \times \prod_{k=1}^{p} (I - rM_{T_k})^{-\gamma_k} (I - M_{T_k})^{\gamma_k}(1) \\ &= \prod_{j>p} (I - M_{T_j})^{\beta_j} \prod_{\substack{j=1 \\ \beta_j \ne 0}}^{p} (I - rM_{T_j})^{-\gamma_j} (I - M_{T_j})^{\beta_j + \gamma_j} \\ &\qquad \times \prod_{\substack{j=1 \\ \beta_j = 0}}^{p} (I - rM_{T_j})^{-\gamma_j} (I - M_{T_j})^{\gamma_j}(1). \end{aligned}$$

Note also that the limit

$$\text{(2.5)} \qquad s\text{-}\lim_{r \to 1^-} \prod_{\substack{j=1 \\ \beta_j = 0}}^{p} (I - rM_{T_j})^{-\gamma_j} (I - M_{T_j})^{\gamma_j}(1)$$

exists, as in the first part of the proof. Since

$$\prod_{\substack{j=1 \\ \beta_j \ne 0}}^{p} (I - M_{T_j})^{\beta_j} \prod_{\substack{j=1 \\ \beta_j = 0}}^{p} (I - M_{T_j})^{\gamma_j} \prod_{j>p} (I - M_{T_j})^{\beta_j}(1) \ge 0$$

by hypothesis, it follows that

$$(2.6)\qquad \prod_{j>p}(I-M_{T_j})^{\beta_j}\times\prod_{\substack{j=1\\ \beta_j\neq 0}}^{p}(I-M_{T_j})^{\beta_j}\times\prod_{\substack{j=1\\ \beta_j=0}}^{p}(I-rM_{T_j})^{-\gamma_j}(I-M_{T_j})^{\gamma_j}(1)\geq 0,$$

by the series expansion of $\prod(I-rM_{T_j})^{-\gamma_j}$ and the fact that the operators M_{T_j} preserve positivity. We have only to note that

$$(2.7)\qquad \lim_{r\to 1^-}(I-rM_{T_j})^{-\gamma_j}(I-M_{T_j})^{\beta_j+\gamma_j}=(I-M_{T_j})^{\beta_j}\quad(\beta_j\neq 0)$$

in the uniform topology, by an easy direct argument valid for Banach space contractions.

Letting $r\to 1^-$ in (2.4), and using (2.5), (2.6) and (2.7), as well as the continuity of the operators M_{T_j} in the strong operator topology of $\mathcal{L}(\mathcal{H})$, we derive (2.2). To obtain the estimate (2.3), note that

$$\prod_{\substack{j=1\\ \beta_j=0}}^{p}(I-M_{T_j})^{\delta_j}(\Delta_T^{\beta})\geq 0$$

for all $\delta_j\leq\gamma_j$, $j\in\{k:\beta_k=0\}$, by hypothesis. Therefore, by virtue of [CuVa, Lemma 3.6],

$$(2.8)\qquad \prod_{\substack{j=1\\ \beta_j=0}}^{p}(I-rM_{T_j})^{-\gamma_j}(I-M_{T_j})^{\gamma_j}(\Delta_T^{\beta})\leq\Delta_T^{\beta}.$$

Since the left-hand side of (2.8) has the same limit as the left-hand side of (2.4), letting $r\to 1^-$ in (2.8), we infer the estimate (2.3), which completes the proof of the lemma. **Λ**

We briefly pause to recall that a contraction $D\in\mathcal{L}(\mathcal{H})$ is said to be of class $C_{0\cdot}$ if $s-\lim_{k\to\infty}D^k=0$ (see [SzFo]).

Lemma 2.2 *Let $\gamma\geq e^{(n)}$, let $T\in\mathcal{L}(\mathcal{H})^n$ be γ-contractive, and let $p\in\{1,\dots,n\}$. Then there are two Hilbert spaces $\mathcal{G}$, $\mathcal{M}$, two γ-contractive c.m. $R\in\mathcal{L}(\mathcal{G})^n$, $Q\in\mathcal{L}(\mathcal{M})^n$, and an isometry $V:\mathcal{H}\to\mathcal{G}\oplus\mathcal{M}$, with the following properties:*

(1) *Given $j=1,\dots,n$,*

$$(2.9)\qquad VT_j=(R_j\oplus Q_j)V;$$

(2) $R_1, \ldots, R_p$ *are of class* $C_{0\cdot}$, *and* $(Q_1, \ldots, Q_p)$ *is a* $\gamma^{(p)}$*-isometry, where* $\gamma^{(p)} := (\gamma_1, \ldots, \gamma_p) \in \mathbf{Z}_+^p$;

(3) *if for some* $k \in \{1, \ldots, n\}$ *the operator* T_k *is an isometry (resp.* $T_k \in C_{0\cdot}$*), then both* R_k, Q_k *are isometries (resp.* R_k, $Q_k \in C_{0\cdot}$*).*

Proof. We define an operator $V_0 : \mathcal{H} \to \ell^2(\mathbf{Z}_+^p, \mathcal{H})$ by the formula

$$(V_0 h)(\alpha) := \left[\rho_{\gamma^{(p)}}(\alpha)\Delta_{T^{(p)}}^{\gamma^{(p)}}\right]^{1/2} T^{(p)\alpha} h \tag{2.10}$$

for all $h \in \mathcal{H}$ and $\alpha \in \mathbf{Z}_+^p$, where $T^{(p)} := (T_1, \ldots, T_p)$. It is easily seen that $V_0^* V_0 = J_0$, where J_0 is the operator (2.1) (see also [CuVa, (3.15), (3.21)]). In particular, $V_0^* V_0 \leq 1$.

We set $\mathcal{G} := \overline{V_0\mathcal{H}}$, $V_1 := (1 - V_0^* V_0)^{1/2}$, and $\mathcal{M} := \overline{V_1\mathcal{H}}$. Note the estimates

$$\begin{cases} T_j^* V_0^* V_0 T_j \leq V_0^* V_0 \\ T_j^* V_1^2 T_j \leq V_1^2 \end{cases} \quad (j = 1, \ldots, n), \tag{2.11}$$

which are particular cases of (2.2) and (2.3), respectively. By virtue of (2.11) we may define the linear mappings

$$\begin{cases} R_j \cdot V_0 h := V_0 T_j h \\ Q_j \cdot V_1 h := V_1 T_j h \end{cases} \quad (h \in \mathcal{H},\ j = 1, \ldots, n), \tag{2.12}$$

which can be continuously extended to the spaces $\mathcal{G}$, $\mathcal{M}$, respectively. We keep the same notation for the extensions. It follows easily from (2.12) that $R := (R_1, \ldots, R_n) \in \mathcal{L}(\mathcal{G})^n$, $Q := (Q_1, \ldots, Q_n) \in \mathcal{L}(\mathcal{M})^n$ are c.m. In addition, if we set $Vh := V_0 h \oplus V_1 h$ for each $h \in \mathcal{H}$, then $V : \mathcal{H} \to \mathcal{G} \oplus \mathcal{M}$ is an isometry satisfying (2.9), via (2.12).

We show now that R is γ-contractive. Let $\beta \in \mathbf{Z}_+^n$ be fixed. We set, for simplicity, $c_{\alpha,\beta} := (-1)^{|a|}\beta![\alpha!(\beta - \alpha)!]^{-1}$ if $\alpha \leq \beta$, and $c_{\alpha,\beta} := 0$ otherwise. Then we have for $\beta \leq \gamma$:

$$\begin{aligned} \langle \Delta_R^\beta V_0 h, V_0 h \rangle &= \sum_{\alpha \geq 0} c_{\alpha,\beta} \|R^\alpha V_0 h\|^2 \\ &= \sum_{\alpha \geq 0} c_{\alpha,\beta} \|V_0 T^\alpha h\|^2 \\ &= \left\langle \sum_{j=1}^n (I - M_{T_j})^{\beta_j} (V_0^* V_0) h, h \right\rangle \geq 0 \end{aligned}$$

for all $h \in \mathcal{H}$, by virtue of (2.2), via (1.1) and (1.3). Hence $\Delta_R^\beta \geq 0$ for all $\beta \leq \gamma$, i.e., R is γ-contractive.

A similar argument (using (2.3) instead of (2.2)) shows that Q is γ-contractive. Since we have already noticed that (2.9) holds, the assertion (i) is established.
To obtain (ii), note that

$$V_0 T_j = S_j^{(\gamma^{(p)})} V_0 \qquad (j = 1, \dots, p), \tag{2.13}$$

which follows as in ([CuVa, (3.22)]) (with $S_j^{(\gamma^{(p)})}$ given by (1.4)). Then, according to [CuVa, Lemma 3.5], the operators $R_j = S_j^{(\gamma^{(p)})}|\mathcal{G}$ $(j = 1, \dots, p)$ are of class $C_{0\cdot}$.

We must also prove that $\Delta_{Q^{(p)}}^{e^{(p)}} = 0$, where $Q^{(p)} := (Q_1, \dots, Q_p)$. Indeed, the equality

$$\prod_{j=1}^{p} (I - M_{T_j})(1 - V_0^* V_0) = 0, \tag{2.14}$$

holds, as a consequence of [CuVa, Lemma 3.10].

Therefore,

$$\begin{aligned} \langle \Delta_{Q^{(p)}}^{e^{(p)}} V_1 h, V_1 h \rangle &= \sum_{\alpha \le e^{(p)}} (-1)^{|\alpha|} \|Q^\alpha V_1 h\|^2 \\ &= \sum_{\alpha \le e^{(p)}} (-1)^{|\alpha|} \|V_1 T^\alpha h\|^2 \\ &= \left\langle \prod_{j=1}^{p} (I - M_{T_j})(1 - V_0^* V_0) h, h \right\rangle = 0 \end{aligned}$$

for all $h \in \mathcal{H}$, by (2.14). (Here we have used the equality $c_{\alpha,\beta} = (-1)^{|\alpha|}$ when $\beta = e^{(p)}$ and $\alpha \le \beta$.) In other words, $Q^{(p)}$ is a $\gamma^{(p)}$-isometry, and so (ii) is also established.

Now, assume T_k is an isometry for some k. Then we have

$$\begin{aligned} (I - M_{T_k})(V_0^* V_0) &= s\text{–}\lim_{r \to 1^-} (I - M_{T_k}) \prod_{j=1}^{p} (I - r M_{T_j})^{-\gamma_j} (I - M_{T_j})^{\gamma_j} (1) \\ &= 0, \end{aligned}$$

via formula (2.1), since $(I - M_{T_k})(1) = 0$. Thus, by (2.12),

$$\|R_k V_0 h\| = \|V_0 T_k h\| = \|V_0 h\|$$

for all $h \in \mathcal{H}$, showing that R_k is an isometry.

Next, observe that

$$(I - M_{T_k})(1 - V_0^* V_0) = 0,$$

whence we infer that Q_k is also an isometry, by a similar argument.

Finally, assume T_k of class $C_{0\cdot}$. Since

$$VT_k^m = (R_k^m \oplus Q_k^m)V$$

for every integer $m \geq 1$, via (2.9), and since V is an isometry, we deduce readily that $R_k^m V_0 h \to 0$ and $Q_k^m - V_1 h \to 0$ as $m \to \infty$, for each $h \in \mathcal{H}$. Then the fact that both R_k, Q_k are contractions, and the definition of the spaces $\mathcal{G}$, $\mathcal{M}$, imply that R_k, Q_k must be of class $C_{0\cdot}$. This establishes (iii), and concludes the proof of the lemma. **Λ**

Lemma 2.3 *Let $\mathcal{H}$, $\mathcal{G}$, $\mathcal{M}$ be Hilbert spaces, let $T \in \mathcal{L}(\mathcal{H})^n$, $R \in \mathcal{L}(\mathcal{G})^n$, $Q \in \mathcal{L}(\mathcal{M})^n$ be c.m., and let $V : \mathcal{H} \to \mathcal{G} \oplus \mathcal{M}$ be an isometry such that $VT_j = (R_j \oplus Q_j)V$ $(j = 1, \ldots, n)$. Then we have*

$$\Delta_T^\beta = V^*(\Delta_R^\beta \oplus \Delta_Q^\beta)V \qquad (\beta \in Z_+^n). \tag{2.15}$$

Proof. Let $c_{\alpha,\beta}$ be as in the proof of Lemma 2.2. We also write $Vh = V_0 h \oplus V_1 h$ for each $h \in \mathcal{H}$. Then we have

$$\begin{aligned} \langle \Delta_T^\beta h, h\rangle &= \sum_{\alpha \geq 0} c_{\alpha,\beta} \|VT^\alpha h\|^2 \\ &= \sum_{\alpha \geq 0} c_{\alpha,\beta}(\|R^\alpha V_0 h\|^2 + \|Q^\alpha V_1 h\|^2) \\ &= \langle \Delta_R^\beta V_0 h, V_0 h\rangle + \langle \Delta_Q^\beta V_1 h, V_1 h\rangle \\ &= \langle V^*(\Delta_R^\beta \oplus \Delta_Q^\beta)Vh, h\rangle \end{aligned}$$

for all $h \in \mathcal{H}$. Hence (2.15) holds. **Λ**

2.1. Proof of Theorem 1.2. (1) $\Rightarrow$ (2) This is a consequence of Lemma 2.2.2, with $p = n$, via (2).

(2) $\Rightarrow$ (1) This follows from Lemma 2.2.3, by virtue of [CuVa, Lemma 3.5]. The proof of the theorem is now complete. **Λ**

For additional results along the lines of Theorem 1.2 the reader is referred to [Vas].

Unlike the spherical isometries studied in [MuVa]), the polydisc isometries, are, in general, not subnormal (see [CuVa, p. 802]). Nevertheless, γ-isometries possess a certain structure which seems to merit further consideration.

Definition 2.4 *Let $\gamma \in \mathbf{Z}_+^n$, $\gamma \geq e^{(n)}$, let $p \in \{1, \ldots, n\}$, and let $T \in \mathcal{L}(\mathcal{H})^n$ be a c.m. We say that T is a (γ, p)-isometry if the following conditions are satisfied:*

(1) T is γ-contractive;
(2) there are p distinct integers $k_1, \ldots, k_p$ in the set $\{1, \ldots, n\}$ such that $(T_{k_1}, \ldots, T_{k_p})$ is a $(\gamma_{k_1}, \ldots, \gamma_{k_p})$-isometry;
(3) If $j \in \{1, \ldots, n\} \setminus \{k_1, \ldots, k_p\}$, then either T_j is an isometry or T_j is of class $C_{0\cdot}$.

Remark 2.5 (1) Every γ-isometry is a (γ, n)-isometry.
(2) If $T \in \mathcal{L}(\mathcal{H})^n$ is a c.m. consisting of isometries, then T is a $(\gamma, 1)$-isometry for all $\gamma \geq e^{(n)}$.
(3) If $T \in \mathcal{L}(\mathcal{H})^n$ is a $(\gamma, 1)$-isometry not of the form in (ii) above, then, without loss of generality, we may suppose that $T = (T_1, \ldots, T_q, T_{q+1}, \ldots, T_n)$, where $1 \leq q \leq n-1$, $T_1, \ldots, T_q$ are of class $C_{0\cdot}$, and $T_{q+1}, \ldots, T_n$ are isometries. We may apply Lemma 2.2.2 to this particular situation (with $p = q$). Note also that the operator (2.10) is in this case an isometry, via [CuVa, Lemma 3.9]. Consequently, T has the form

$$(S_1^{(\gamma^{(q)})}|\mathcal{G}\ , \ldots , \ S_q^{(\gamma^{(q)})}|\mathcal{G}\ , R_{q+1}\ , \ldots , \ R_n), \tag{2.16}$$

modulo unitary equivalence, where $\mathcal{G} = V_0\mathcal{H}$ (V_0 given by (2.10)), and $R_{q+1}, \ldots, R_n$ are isometries on $\mathcal{G}$, by virtue of Lemma 2.2.2.

Our goal is to describe the structure of an arbitrary γ-isometry in terms of $(\gamma, 1)$-isometries. We need two more technical lemmas.

Lemma 2.6 *Let $T \in \mathcal{L}(\mathcal{H})^n$ be a (γ, p)-isometry, with $p \geq 2$. Then there exist Hilbert spaces $\mathcal{G}_k$, c.m. $R^{(k)} \in \mathcal{L}(\mathcal{G}_k)^n$ ($k = 1, 2, 3$), and an isometry $V : \mathcal{H} \to \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3$ such that*

$$VT_j = (R_j^{(1)} \oplus R_j^{(2)} \oplus R_j^{(3)})V \quad (j = 1, ..., n),$$

where $R^{(1)}$ is a $(\gamma, 1)$-isometry, and $R^{(2)}, R^{(3)}$ are $(\gamma, p-1)$-isometries.

Proof. Without loss of generality we may assume that $T = (T_1, \ldots, T_p, T_{p+1}, \ldots, T_q, T_{q+1}, \ \ldots, T_n)$, where $T^{(p)} := (T_1, \ldots, T_p)$ is a $\gamma^{(p)}$-isometry ($\gamma^{(p)} := (\gamma_1, \ldots, \gamma_p)$), $T_{p+1}, \ldots, T_q$ are isometries, and $T_{q+1}, \ldots, T_n$ are of class $C_{0\cdot}$ (of course, the last two kinds of operators may be absent). We shall apply Lemma 2.2.2 to γ, T, and $p-1$. Let V_0' be the operator given by (2.10). We also set $\mathcal{G}_1 := \overline{V_0'\mathcal{H}}$, $V_1' := (1 - V_0'^* V_0')^{1/2}$, $\mathcal{H}' := \overline{V_1'\mathcal{H}}$, and $V'h := V_0'h \oplus V_1'h$ ($h \in \mathcal{H}$).

According to Lemma 2.2.2, there exist c.m. $R^{(1)} \in \mathcal{L}(\mathcal{G}^{(1)})^n$, $T' \in \mathcal{L}(\mathcal{H}')^n$ which are γ-contractive, such that $V'T_j = (R_j^{(1)} \oplus T_j')V'$ $(j = 1, \ldots, n)$. Moreover, $R_1^{(1)}, \ldots, R_{p-1}^{(1)}$ are of class $C_{0\cdot}$, and $(T_1', \ldots, T_{p-1}')$ is a $\gamma^{(p-1)}$-isometry. It also follows from Lemma 2.2.2 that $R_{p+1}^{(1)}, \ldots, R_q^{(1)}, T_{p+1}', \ldots, T_q'$ are isometries, and that $R_{q+1}^{(1)}, \ldots, R_n^{(1)}, T_{q+1}', \ldots, T_n'$ are of class $C_{0\cdot}$.

Let us show now that $R_p^{(1)}$ is an isometry. Indeed, since $T^{(p)}$ is a $\gamma^{(p)}$-isometry, we have

$$(I - M_{T_p})(V_0'^* V_0') = s\text{-}\lim_{r \to 1^-} \prod_{j=1}^{p-1} (I - rM_{T_j})^{-\gamma_j} (I - M_{T_j})^{\gamma_j} (I - M_{T_p})(1) = 0.$$

Hence $\|R_p^{(1)} V_0' h\| = \|V_0' T_p h\| = \|V_0' h\|$ for all $h \in \mathcal{H}$, via (2.12) and the remark above, i.e., $R_p^{(1)}$ is an isometry.

Consequently, $R^{(1)}$ is actually a $(\gamma, 1)$-isometry. We have to deal now with the c.m. $T' \in \mathcal{L}(\mathcal{H}')^n$, in which T_p' is simply a contraction. We shall apply Lemma 2.2.2 to γ, T'', 1, where

$$T'' := (T_p', T_1', \ldots, T_{p-1}', T_{p+1}', \ldots, T_n').$$

According to Lemma 2.2.2, there are two Hilbert spaces $\mathcal{G}_2$,$\mathcal{G}_3$, two γ-contractive c.m. $R^{(2)} \in \mathcal{L}(\mathcal{G}_2)^n$, $R^{(3)} \in \mathcal{L}(\mathcal{G}_3)^n$, and an isometry $V'' : \mathcal{H}' \to \mathcal{G}_2 \oplus \mathcal{G}_3$ such that $V''T_j' = (R_j^{(2)} \oplus R_j^{(3)})V''$ $(j = 1, \ldots, n)$. Note that $(R_1^{(2)}, \ldots, R_{p-1}^{(2)})$, $(R_1^{(3)}, \ldots, R_{p-1}^{(3)})$ are $\gamma^{(p-1)}$-isometries, by Lemma 2.2.3 and the corresponding property of $(T_1', \ldots, T_{p-1}')$. We also have that $R_{p+1}^{(2)}, \ldots, R_q^{(2)}$, $R_{p+1}^{(3)}, \ldots, R_q^{(3)}$ are isometries, and that $R_{q+1}^{(2)}, \ldots, R_n^{(2)}$, $R_{q+1}^{(3)}, \ldots, R_n^{(3)}$ are of class $C_{0\cdot}$, by Lemma 2.2.2 and the corresponding properties of $T_{p+1}', \ldots, T_q'$, resp. $T_{q+1}', \ldots, T_n'$. Finally, it follows from Lemma 2.2.2 that $R_p^{(2)}$ is of class $C_{0\cdot}$, and $R_p^{(3)}$ is an isometry. Therefore both $R^{(2)}$, $R^{(3)}$ are $(\gamma, p-1)$-isometries. We have only to note that $V : \mathcal{H} \to \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3$, given by $V := (1 \oplus V'')V'$, is the required isometry, via the properties of V', V''. This completes the proof of the lemma. $\boldsymbol{\Lambda}$

Lemma 2.7 *Let $T \in \mathcal{L}(\mathcal{H})^n$ be a (γ, p)-isometry with $T^{(p)} := (T_1, \ldots, T_p)$ a $\gamma^{(p)}$-isometry, $\gamma^{(p)} := (\gamma_1, \ldots, \gamma_p)$. Then there are Hilbert spaces $\mathcal{G}_F$, γ-contractive c.m. $R^F \in L(\mathcal{G}_F)$ $(\emptyset \neq F \subseteq \{1, \ldots, p\})$ and an isometry*

$$V : \mathcal{H} \to \bigoplus_{\emptyset \neq F \subseteq \{1, \ldots, p\}} \mathcal{G}_F \tag{2.17}$$

with the following properties:

(1) *for all* $j = 1, \ldots, n$,

$$VT_j = \Big(\bigoplus_{\emptyset \neq F \subseteq \{1,\ldots,p\}} R_j^F \Big) V . \tag{2.18}$$

(2) R^F *is an isometry if* $j \in F$, *and* R^F *is of class* $C_{0\cdot}$ *if* $j \in \{1, \ldots, p\} \setminus F$.
(3) R^F *is an isometry (resp. of class* $C_{0\cdot}$*) whenever* T_j *is an isometry (resp. of class* $C_{0\cdot}$*) for all* $j \geq p+1$.

Proof. We prove the assertion by induction with respect to $p \geq 1$, for an arbitrary $n \geq p$. If $p = 1$, T_1 is an isometry; moreover, T_j is either an isometry or of class $C_{0\cdot}$ for all $j \geq 2$. Hence the property holds with $\mathcal{G}_{\{1\}} = \mathcal{H}$ and V the identity on $\mathcal{H}$.

Now, assume that the assertion holds for $p-1$ ($p \geq 2$), and let us prove it for p. Let T be as in the statement of the lemma. According to (the proof of) Lemma 2.2.6, we can find Hilbert spaces $\mathcal{G}_p$, $\mathcal{H}^{(1)}$, $\mathcal{H}^{(2)}$, and γ-contractive c.m. $R^{(p)} \in \mathcal{L}(\mathcal{G}_p)^n$, $Z^{(k)} \in \mathcal{L}(\mathcal{H}^{(k)})^n$ ($k = 1, 2$) with the following properties:

(1) $R_1^{(p)}, \ldots, R_{p-1}^{(p)}$ are of class $C_{0\cdot}$, and $R_p^{(p)}$ is an isometry;
(2) $(Z_1^{(k)}, \ldots, Z_{p-1}^{(k)})$ are $\gamma^{(p-1)}$-isometries $(k = 1,)2$, $Z_p^{(1)}$ is of class $C_{0\cdot}$, and $Z_p^{(2)}$ is an isometry;
(3) $R_j^{(p)}$, $Z_j^{(k)}$ are isometries (resp. of class $C_{0\cdot}$) whenever T_j is an isometry (resp. of class $C_{0\cdot}$) for all $j \geq p+1$, $k = 1, 2$;
(4) there is an isometry

$$V^{(0)} : \mathcal{H} \to \mathcal{G}_p \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \tag{2.19}$$

such that $V^{(0)} T_j = (R_j^{(p)} \oplus Z_j^{(1)} \oplus Z_j^{(2)}) V^{(0)}$ for all $j = 1, \ldots, n$. (Note that $(\mathcal{G}_p, R^{(p)})$ is the pair needed in (2.17), (2.18) for $F = \{p\}$.)

By the induction hypothesis, there are Hilbert spaces $\mathcal{G}_{k,J}$, γ-contractive c.m. $R^{k,J} \in \mathcal{L}(\mathcal{G}_{k,J})$ ($\emptyset \neq J \subseteq \{1, \ldots, p-1\}$) and isometries

$$V^{(k)} : \mathcal{H}^{(k)} \to \bigoplus_{\emptyset \neq J \subseteq \{1,\ldots,p-1\}} \mathcal{G}_{k,J} \tag{2.20}$$

such that

$$V^{(k)} Z_j^{(k)} = \Big(\bigoplus_{\emptyset \neq J \subseteq \{1,\ldots,p-1\}} R_j^{k,J} \Big) V^{(k)}$$

for all $j = 1, \ldots, n$ and $k = 1, 2$. Moreover, the $R_j^{k,J}$ are isometries if $j \in J$ or $j \geq p$, and $Z_j^{(k)}$ is an isometry, and they are of class $C_{0\cdot}$ otherwise.

Let $F \subseteq \{1, \ldots, p\}$, $F \neq \emptyset$. Then we set

$$\mathcal{G}_F := \begin{cases} \mathcal{G}_p & \text{if } F = \{p\}, \\ \mathcal{G}_{1,F} & \text{if } p \notin F, \\ \mathcal{G}_{2,F\setminus\{p\}} & \text{if } p \in F,\ F \neq \{p\}. \end{cases}$$

We also define

$$R^F := \begin{cases} R^{(p)} & \text{if } F = \{p\}, \\ R^{1,F} & \text{if } p \notin F, \\ R^{2,F\setminus\{p\}} & \text{if } p \in F,\ F \neq \{p\}. \end{cases}$$

Note that R_j^F is an isometry if $j \in F$, and of class $C_{0\cdot}$ if $j \in \{1,\dots,p\}\backslash F$. The isometry V required for (2.18) is now easily obtained from (2.19) and (2.20). The proof of the lemma is complete. $\mathbf{\Lambda}$

Lemma 2.2.7 shows, in particular, that every γ-isometry is unitarily equivalent to a (finite) direct sum of $(\gamma, 1)$-isometries restricted to invariant subspaces.

Combining Theorem 1.1.2 and Lemma 2.2.7, we derive the following structure result.

Theorem 2.8 *Let $\gamma \in \mathbf{Z}_+^n$, $\gamma \geq e^{(n)}$, and let $T \in \mathcal{L}(\mathcal{H})^n$ be γ-contractive. Then there exist Hilbert spaces $\mathcal{G}_F$, γ-contractive c.m. $R^F \in \mathcal{L}(\mathcal{G}_F)^n$ ($F \subseteq \{1,\dots,n\}$), and an isometry*

$$V : \mathcal{H} \to \bigoplus_{F\subseteq\{1,\dots,n\}} \mathcal{G}_F \tag{2.21}$$

with the following properties:

(1) *For all $j = 1,\dots,n$,*

$$VT_j = \Big(\bigoplus_{F\subseteq\{1,\dots,n\}} R_j^F\Big)V; \tag{2.22}$$

(2) *$R_1^{\varnothing},\dots,R_n^{\varnothing}$ are operators of class $C_{0\cdot}$, and for every $F \subseteq \{1,\dots,n\}$, $F \neq \varnothing$, the operators R_j^F are isometries if $j \in F$, and are of class $C_{0\cdot}$ if $j \in \{1,\dots,n\} \setminus F$.*

Proof. According to Theorem 1.1.2, there are Hilbert spaces $\mathcal{G}_0$, $\mathcal{H}'$, γ-contractive c.m. $R^{(0)} \in \mathcal{L}(\mathcal{G}_0)^n$, $T' \in \mathcal{L}(\mathcal{H}')^n$, and an isometry $V' : \mathcal{H} \to \mathcal{G}_0 \oplus \mathcal{H}'$ such that $V'T_j = (R_j^{(0)} \oplus T_j')V'$ ($j = 1,\dots n$). Moreover, $R^{(0)}$ consists of operators of class $C_{0\cdot}$ (via ([CuVa, Theorem 3.16]), and T' is a γ-isometry. To complete the proof of the theorem, we define $\mathcal{G}_{\varnothing} := \mathcal{G}_0$, $R^{\varnothing} := R^{(0)}$, and we apply Lemma 2.2.7 (with $p = n$) to T'. From (2.17) and (2.18) written for T', as well as using $R^{\emptyset}$, $\mathcal{G}_{\emptyset}$ and V', we infer readily (2.21) and (2.22), which concludes the proof. $\boldsymbol{\Lambda}$

Let us remark that if the c.m. T$\in \mathcal{L}(\mathcal{H})^n$ satisfies (2.22), with all R^F γ-contractive, then T is also γ-contractive by virtue of (an extended version of) Lemma 2.3.

3. Some related results Theorem 2.8 shows that the structure of a c.m. consisting of contractions, even if some positivity conditions are satisfied, is in general rather complicated. Unlike the case associated with the geometry of the unit ball (see [MuVa]), the case associated with the polydisc is unexpectedly intricate. Besides the standard model (1.4) or those c.m. consisting of isometries, which can be regarded as "extreme" cases, there also occur "mixed" cases. We refer here to $(\gamma, 1)$-isometries $T \in \mathcal{L}(\mathcal{H})^n$ $(n \geq 2)$ whose form (modulo a permutation of indices) is $T = (T_1, \ldots, T_q, \ldots, T_n)$, where $T_1, \ldots, T_q$ are operators of class $C_{0\cdot}$, and $T_{q+1}, \ldots, T_n$ are isometries $(1 \leq q \leq n-1)$, as in Remark 2.5(iii). Since for the extreme cases more information is available, we think it is useful to give a version of Theorem 2.2.8 in which the "mixed" c.m. are automatically eliminated.

Proposition 3.1 *Let $\gamma \geq e^{(n)}$ and let $T \in \mathcal{L}(\mathcal{H})^n$ be a c.m. The following conditions are equivalent.*

(a) *T is γ–contractive, and*

(3.1)
$$(I - M_{T_j})(1 - J_0) = 0 \quad (j = 1, \ldots, n),$$

where

(3.2)
$$J_0 := s - \lim_{r \to 1^-} \prod_{j=1}^{n} (I - rM_{T_j})^{-\gamma_j}(\Delta_T^{\gamma}).$$

(b) *T is unitarily equivalent to the restriction of $S^{(\gamma)} \oplus Q$ to an invariant subspace, where Q is a c.m. consisting of isometries.*

Proof. (a) $\Rightarrow$ (b) We follow the line of the proof of Theorem 1.1.2 (or, rather, that of Lemma 2.2.2). We have the equality (2.9), in which R may be replaced by $S^{(\gamma)}$ (via (2.12) and (2.13)), and Q is a γ-isometry. We have only to show that $Q_1, \ldots, Q_n$ are actually isometries.

Indeed, since J_0 given by (3.2) coincides with J_0 given by (2.1) (for $p = n$), we have, using the notation of Lemma 2.2.2, $V_0^* V_0 = J_0$, and $V_1^2 = 1 - J_0$. Since (3.1) can be rewritten as

$$T_j^* V_1^2 T_j = V_1^2 \quad (j = 1, \ldots, n),$$

it follows from (2.12) that $Q_1, \ldots, Q_n$ are isometries.

(b) $\Rightarrow$ (a) It follows from (b) that there are Hilbert spaces $\mathcal{G} \subseteq \ell^2(\mathbf{Z}_+^n, \mathcal{H})$ and $\mathcal{M}$, and an isometry $V : \mathcal{H} \to \mathcal{G} \oplus \mathcal{M}$ such that $VT_j = (S_j^{(\gamma)} \oplus Q_j)V$ $(j = 1, \ldots, n)$. Let $R_j := S_j^{(\gamma)}|\mathcal{G}$ for all j, and $R := (R_1, \ldots, R_n)$.

Note that

$$\sum_{\alpha \geq 0} \rho_\gamma(\alpha) \|(\Delta_R^\gamma)^{1/2} R^\alpha g\|^2 = \|g\|^2 \quad (g \in \mathcal{G}), \tag{3.3}$$

by [CuVa, Lemma 3.5, (3.21) and (3.23)]. Then we have

$$\begin{aligned} J_0 h &= \sum_{\alpha \geq 0} \rho_\gamma(\alpha) T^{*\alpha} \Delta_T^\gamma T^\alpha h \\ &= \sum_{\alpha \geq 0} \rho_\gamma(\alpha) T^{*\alpha} V^* (\Delta_R^\gamma \oplus \Delta_Q^\gamma) V T^\alpha h \\ &= V^* \Big[\Big(\sum_{\alpha \geq 0} \rho_\gamma(\alpha) R^{*\alpha} \Delta_R^\gamma R^\alpha \Big) \oplus 0 \Big] V h \\ &= V^*(1 \oplus 0) V h \end{aligned}$$

for all $h \in \mathcal{H}$, by [CuVa, (3.15)], Lemma 2.2.3, (3.3), and since $\Delta_Q^\gamma = 0$.

Therefore $1 - J_0 = V^*(0 \oplus 1)V$, whence

$$\begin{aligned} (I - M_{T_j})(V^*(0 \oplus 1)V) &= V^*(0 \oplus 1)V - T_j^* V^*(0 \oplus 1) V T_j \\ &= V^*(0 \oplus 1)V - V^*(R_j^* \oplus Q_j^*)(0 \oplus 1)(R_j \oplus Q_j)V \\ &= 0 \end{aligned}$$

for all j, since each Q_j is an isometry. Since T is clearly γ-contractive, this establishes the implication (b) $\Rightarrow$ (a), which concludes the proof of the proposition. **Λ**

An important particular case of all previous assertions is obtained when $\gamma = e := e^{(n)}$. According to Definition 1.1, a c.m. $T \in \mathcal{L}(\mathcal{H})^n$ is e-contractive if $\Delta_T^\alpha \geq 0$ for all $\alpha \leq e$. This is precisely Brehmer's condition, which is equivalent to the existence of a regular unitary dilation (see [Bre] or [SzFo]; see also [CuVa, Section 4]).

The standard model $S \equiv S^{(e)}$ defined via (1.4), becomes

$$(S_j f)(\alpha) = f(\alpha + e_j) \quad (f \in \mathcal{K},\ \alpha \in \mathbf{Z}_+^n,\ j = 1, \ldots, n) \tag{3.4}$$

(recall that $\mathcal{K} = \ell^2(\mathbf{Z}_+^n, \mathcal{H})$ and observe that $\rho_e(\alpha) \equiv 1$). Since

$$\begin{aligned} (S_j^* f)(\alpha) &= f(\alpha - e_j) & &\text{if } \alpha_j \geq 1, \\ &= 0 & &\text{if } \alpha_j = 0 \end{aligned} \tag{3.5}$$

for all $f \in \mathcal{K}$, $\alpha \in \mathbf{Z}_+^n$, $j = 1, \ldots, n$, a simple computation shows that

$$(S_j^* S_k f)(\alpha) = (S_k S_j^* f)(\alpha) = \begin{cases} f(\alpha + e_k - e_j) & \text{if } \alpha_j \geq 1, \\ 0 & \text{if } \alpha_j = 0, \end{cases}$$

whenever $j \neq k$. In other words, the c.m. S is doubly commuting. From this observation we derive the equality

$$S^{*\alpha} S^{\beta} = S^{\beta} S^{*\alpha} \tag{3.6}$$

valid for all α, $\beta \in \mathbf{Z}_+^n$ with $\alpha \circ \beta = 0$, where $\alpha \circ \beta := (\alpha_1\beta_1, \ldots, \alpha_n\beta_n)$.

Now, let $\mathcal{L} := \ell^2(\mathbf{Z}^n, \mathcal{H})$. The space $\mathcal{K}$ can be naturally embedded into $\mathcal{L}$ via the isometry $\mathcal{K} \ni f \to \tilde{f} \in \mathcal{L}$, where $\tilde{f}(\alpha) = f(\alpha)$ if $\alpha \in \mathbf{Z}_+^n$, and $\tilde{f}(\alpha) = 0$ otherwise.

The counterparts of (3.4) and (3.5) on $\mathcal{L}$ are, respectively,

$$(U_j g)(\alpha) = g(\alpha + e_j) \tag{3.7}$$

and

$$(U_j^* g)(\alpha) = g(\alpha - e_j), \tag{3.8}$$

for all $g \in \mathcal{L}$, $\alpha \in \mathbf{Z}^n$, $j = 1, \ldots, n$. Note that $U := (U_1, \ldots, U_n)$ is a c.m. on $\mathcal{L}$ consisting of *unitary* operators. Moreover, we have

$$U_j^{*\alpha} \tilde{f} = (S_j^* f)\tilde{} \quad (f \in K, j = 1, \ldots, n),$$

as one can easily check. Therefore

$$U^{*\alpha} \tilde{f} = (S^{*\alpha} f)\tilde{} \quad (\alpha \in \mathbf{Z}_+^n,\ f \in \mathcal{K}). \tag{3.9}$$

As we have already mentioned, if $T \in \mathcal{L}(\mathcal{H})^n$ is e-contractive, then T has a regular unitary dilation (see, for instance, [SzFo, Theorem I.9.1]). In other words, there is a Hilbert space $\mathcal{R}$, an isometry $W : \mathcal{H} \to \mathcal{R}$ and a c.m. $D \in \mathcal{L}(\mathcal{R})^n$ consisting of unitary operators such that

$$W^* D^{*\alpha} D^{\beta} W = T^{*\alpha} T^{\beta} \tag{3.10}$$

for all α, $\beta \in \mathbf{Z}_+^*$ with $\alpha \circ \beta = 0$. For brevity, we shall say that $(\mathcal{R}, W, D)$ is a r.u.d. for T.

We shall show that our methods provide, in particular, a new proof of the existence of a regular unitary dilation for every e-contractive c.m. ([SzFo, Theorem I.9.1]). In the remaining part of this section we shall discuss this question.

Lemma 3.2 *Let $T \in \mathcal{L}(\mathcal{H})^n$ be e-contractive. If $s\text{–}\lim_{k\to\infty} T_j^k = 0$ $(j = 1, \ldots, n)$, then T has a regular unitary dilation.*

Proof. We keep the notation above. By virtue of [CuVa, Theorem 3.16], we may assume without loss of generality that $T_j = S_j|\mathcal{G}$, where $\mathcal{G} \subseteq \mathcal{K}$ is invariant under S_j $(j = 1, \dots, n)$. We shall show that $(\mathcal{L}, W, U)$ is a r.u.d. for T.

Indeed, let $W : \mathcal{G} \to \mathcal{L}$ be the isometry $Wg := \tilde{g}$, and let α, $\beta \in \mathbf{Z}_+^*$ be such that $\alpha \circ \beta = 0$. Then for all g_1, $g_2 \in \mathcal{G}$ we have

$$\begin{aligned} \left\langle W^* U^{*\alpha} U^\beta W g_1, g_2 \right\rangle &= \left\langle U^{*\alpha} \tilde{g}_1, U^{*\beta} \tilde{g}_2 \right\rangle \\ &= \left\langle S^{*\alpha} g_1, S^{*\beta} g_2 \right\rangle = \left\langle S^{*\alpha} S^\beta g_1, g_2 \right\rangle \\ &= \left\langle T^{*\alpha} T^\beta g_1, g_2 \right\rangle, \end{aligned}$$

by (3.9), (3.6), and the invariance of $\mathcal{G}$ under $S_1, \dots, S_n$. This shows that (3.10) holds, i.e., $(\mathcal{L}, W, U)$ is a r.u.d. for T. $\mathbf{\Lambda}$

Lemma 3.3 *Let $T \in \mathcal{L}(\mathcal{H})^n$ be a c.m. consisting of isometries. Then T has a regular unitary dilation.*

Proof. According to [SzFo, Proposition I.6.2], there is a Hilbert space $\mathcal{R}$, an isometry $V : \mathcal{H} \to \mathcal{R}$, and a c.m. $D \in \mathcal{L}(\mathcal{R})^n$ consisting of unitary operators such that $D_j V = V T_j$ for all j. It is an easy matter to prove that

$$V^* D^{*\alpha} D^\beta V = T^{*\alpha} T^\beta \tag{3.11}$$

for all α, $\beta \in Z_+^n$. Thus, $(\mathcal{R}, V, D)$ is a r.u.d. for T. $\mathbf{\Lambda}$

Remark 3.4 *Let T, W, D be as in* (3.10). *If $\alpha \in \mathbf{Z}^n$, then α can be uniquely written as $\alpha = \alpha^+ - \alpha^-$, with α^-, $\alpha^+ \in \mathbf{Z}_+^n$, where $\alpha_j^+ := \max\{\alpha_j, 0\}$, $\alpha_j^- = \max\{-\alpha_j, 0\}$ for all j. Then* (3.10) *can be rewritten as*

$$W^* D^\alpha W = T^{*\alpha^-} T^{\alpha^+} \tag{3.12}$$

for all $\alpha \in \mathbf{Z}^n$, where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ makes sense since $D_1, \dots, D_n$ are unitary.

Lemma 3.5 *Let $T \in \mathcal{L}(\mathcal{H})^n$ $(n \geq 2)$ be an $(e,1)$-isometry of the form $T = (T_1, \dots, T_p, \dots T_n)$ $(1 \leq p \leq n-1)$, where $T_1, \dots, T_p$ are of class $C_{0\cdot}$, and $T_{p+1}, \dots, T_n$ are isometries. Then T has a regular unitary dilation.*

Proof. According to [SzFo, Section I.9], it is sufficient to show that for every function $f : \mathbf{Z}_+^n \to \mathcal{H}$ with finite support, and all α, $\beta \in \mathbf{Z}_+^n$, we have

$$\sum_{\alpha,\beta} \left\langle T^{*(\alpha-\beta)^-} T^{(\alpha-\beta)^+} f(\alpha), f(\beta) \right\rangle \geq 0. \tag{3.13}$$

In order to show that (3.13) holds, we shall use the corresponding properties of the c.m. $T' := (T_1, \ldots, T_p)$, $T'' := (T_{p+1}, \ldots T_n)$. For every $\alpha \in \mathbf{Z}_+^n$ we also let $\alpha' := (\alpha_1, \ldots, \alpha_p) \in \mathbf{Z}^p$, $\alpha'' := (\alpha_{p+1}, \ldots, \alpha_n) \in \mathbf{Z}^{n-p}$.

First of all notice that

$$T^{*(\alpha-\beta)^-}T^{(\alpha-\beta)^+} = T'^{*(\alpha'-\beta')^-}T''^{*(\alpha''-\beta)^-}T'^{(\alpha'-\beta')^+}T''^{(\alpha''-\beta'')^+} \tag{3.14}$$

for all α, $\beta \in \mathbf{Z}_+^n$.

From the proof of Lemma 3.3 (and without loss of generality) it follows that there exist a Hilbert space $\mathcal{R}'' \supseteq \mathcal{H}$, and a c.m. $U'' \in \mathcal{L}(\mathcal{R}'')^n$ consisting of unitary operators, such that $U_j''|\mathcal{H} = T_j''$ $(j = p+1, \ldots, n)$. Then, with f as in (3.13), we have

$$\begin{aligned} & \left\langle T^{*(\alpha-\beta)^-}T^{(\alpha-\beta)^+}f(\alpha), f(\beta)\right\rangle \\ = & \left\langle U''^{*\beta''}U''^{\alpha''}T'^{(\alpha'-\beta')^+}f(\alpha), T'^{(\alpha'-\beta')^-}f(\beta)\right\rangle \\ = & \left\langle T'^{*(\alpha'-\beta')^-}T'^{(\alpha'-\beta')^+}T''^{\alpha''}f(\alpha), T''^{\beta''}f(\beta)\right\rangle, \end{aligned} \tag{3.15}$$

by (3.14) and (3.11) (and since U'' is an extension of T'').

Now, let $U' \in \mathcal{L}(\mathcal{R}')^n$ be a regular unitary dilation of T', which exists by Lemma 3.2. We may assume $\mathcal{R}' \supseteq \mathcal{H}$. Then we can write, via (3.12),

$$\begin{aligned} & \left\langle T'^{*(\alpha'-\beta')^-}T'^{(\alpha'-\beta')^+}T''^{\alpha''}f(\alpha), T''^{\beta''}f(\beta)\right\rangle \\ = & \left\langle U'^{*\beta'}U'^{\alpha'}T''^{\alpha'}f(\alpha), T''^{\beta''}f(\beta)\right\rangle. \end{aligned} \tag{3.16}$$

But we have

$$\sum_{\alpha,\beta}\left\langle U'^{\alpha'}T''^{\alpha''}f(\alpha), T''^{\beta''}f(\beta)\right\rangle = \left\|\sum_{\alpha}U'^{\alpha'}T''^{\alpha''}f(\alpha)\right\|^2 \geq 0; \tag{3.17}$$

consequently, (3.13) holds since the left-hand side of (3.13) coincides with the left-hand side of (3.17), via (3.14)–(3.16). This concludes the proof of the lemma. **Λ**

Although our proof of Lemma 3.3.5 needs Naimark's dilation theorem (see [SzFo, Theorem I.7.1]), the hypothesis therein is easily verified in this particular case.

Lemma 3.6 *Let $T \in \mathcal{L}(\mathcal{H})^n$, $R^{(k)} \in \mathcal{L}(\mathcal{H}^{(k)})^n$ be c.m. $(k = 1, \ldots, m)$, and let $V : \mathcal{H} \to \mathcal{H}^{(1)} \oplus \cdots \oplus \mathcal{H}^{(m)}$ be an isometry such that*

$$VT_j = (R_j^{(1)} \oplus \cdots \oplus R_j^{(m)})V \quad (j = 1, \ldots, n). \tag{3.18}$$

If every $R^{(k)}$ has a regular unitary dilation, then T has a regular unitary dilation.

Proof. Let $(\mathcal{R}^{(k)}, W_k, D^{(k)})$ be a r.u.d. for $R^{(k)}$ $(k = 1, \dots, m)$. Define $\mathcal{R} := \mathcal{R}^{(1)} \oplus \cdots \oplus \mathcal{R}^{(m)}$ and $D_j := D_j^{(1)} \oplus \cdots \oplus D_j^{(m)}$ $(j = 1, \dots, n)$, and let $V_k : \mathcal{H} \to \mathcal{H}^{(k)}$ be given by

$$Vh = V_1 h \oplus \cdots \oplus V_m h \quad (h \in \mathcal{H}).$$

Note that

$$V_k T_j = R_j^{(k)} V_k k \quad (j = 1, \dots, n, \; k = 1, \dots, m). \tag{3.19}$$

which follows from (3.18). Let $W : \mathcal{H}^{(1)} \oplus \cdots \oplus \mathcal{H}^{(m)} \to \mathcal{R}$ be the isometry $W : W_1 \oplus \cdots \oplus W_n$. Then $WV : \mathcal{H} \to \mathcal{R}$ is an isometry, and for all α, $\beta \in \mathbf{Z}_+^n$ with $\alpha \circ \beta = 0$, and all h_1, $h_2 \in \mathcal{H}$ we have

$$\begin{aligned}
\left\langle V^* W^* D^{*\alpha} D^\beta W V h_{1,} h_2 \right\rangle &= \left\langle \bigoplus_{k=1}^m D^{(k)\beta} W_k V_k h_1, \bigoplus_{k=1}^m D^{(k)\alpha} W_k V_k h_2 \right\rangle \\
&= \sum_{k=1}^m \left\langle W_k^* D^{(k)*\alpha} D^{(k)\beta} W_k V_k h_1, V_k h_2 \right\rangle \\
&= \sum_{k=1}^m \left\langle R^{(k)*\alpha} R^{(k)\beta} V_k h_1, V_k h_2 \right\rangle \\
&= \sum_{k=1}^m \left\langle V_k T^\beta h_1, V_k T^\alpha h_2 \right\rangle \\
&= \left\langle V T^\beta h_1, V T^\alpha h_2 \right\rangle \\
&= \left\langle T^{*\beta} T^\alpha h_1, h_2 \right\rangle
\end{aligned}$$

by (3.19) and the fact that $(\mathcal{R}^{(k)}, W_k, D^{(k)})$ is a r.u.d. for $R^{(k)}$. Thus, $(\mathcal{R}, W, D)$ is a r.u.d. for T. $\mathbf{\Lambda}$

We can give now a new proof of the following result (see also [SzFo, Theorem I.9.1]).

Theorem 3.7 *Let $T \in \mathcal{L}(\mathcal{H})^n$ be e-contractive. Then T has a regular unitary dilation.*

Proof. The assertion follows from Theorem 2.8 and Lemmas 3.2, 3.3, and 3.5. $\mathbf{\Lambda}$

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