GENERALIZED BROWDER'S AND WEYL'S THEOREMS FOR BANACH SPACE OPERATORS

RAÚL E. CURTO AND YOUNG MIN HAN

Abstract. We find necessary and sufficient conditions for a Banach space operator T to satisfy the generalized Browder's theorem. We also prove that the spectral mapping theorem holds for the Drazin spectrum and for analytic functions on an open neighborhood of $\sigma(T)$. As applications, we show that if T is algebraically M-hyponormal, or if T is algebraically paranormal, then the generalized Weyl's theorem holds for $f(T)$, where $f \in H((T))$, the space of functions analytic on an open neighborhood of $\sigma(T)$. We also show that if T is reduced by each of its eigenspaces, then the generalized Browder's theorem holds for $f(T)$, for each $f \in H(\sigma(T))$.

1. Introduction

In [24], H. Weyl proved, for hermitian operators on Hilbert space, his celebrated theorem on the structure of the spectrum (Equation (1.1) below). Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators ([12]), and to several classes of operators including seminormal operators ([3], [4]). Recently, M. Berkani and J.J. Koliha [9] introduced the concepts of generalized Weyl's theorem and generalized Browder's theorem, and they showed that T satisfies the generalized Weyl's theorem whenever T is a normal operator on Hilbert space. More recently, M. Berkani and A. Arroud [8] extended this result to hyponormal operators.

In this paper we extend this result to several classes much larger than that of normal operators. We first find necessary and sufficient conditions for a Banach space operator T to satisfy the generalized Browder's theorem (Theorem 2.1). We then characterize the smaller class of operators satisfying the generalized Weyl's theorem (Theorem 2.4). Along the way we prove that the spectral mapping theorem always holds for the Drazin spectrum and for analytic functions on an open neighborhood of $\sigma(T)$ (Theorem 2.7). We have three main applications of our results: if T is algebraically M-hyponormal, or if T is algebraically paranormal, then the generalized Weyl's theorem holds for $f(T)$, for each $f \in H(\sigma(T))$, the space of functions analytic on an open neighborhood of $\sigma(T)$ (Theorems 4.7 and 4.14, respectively); and if T is reduced by each of its eigenspaces, then the generalized Browder's theorem holds for $f(T)$, for each $f \in H(\sigma(T))$ (Corollary 3.5).

As we shall see below, the concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. Let $\mathcal{A}$ be a unital algebra. We say that $x \in \mathcal{A}$ is Drazin invertible of degree k if there exists an element $a \in \mathcal{A}$ such that

$$x^k ax = x^k, \quad axa = a, \quad \text{and} \quad xa = ax.$$

For $a \in \mathcal{A}$, the Drazin spectrum is defined as

$$\sigma_D(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Drazin invertible}\}.$$

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In the case of $T \in \mathcal{B}(\mathcal{X})$, it is well known that T is Drazin invertible if and only if T has finite ascent and descent, which is also equivalent to having T decomposed as $T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent.

Throughout this note let $\mathcal{B}(\mathcal{X})$, $\mathcal{B}_0(\mathcal{X})$ and $\mathcal{B}_{00}(\mathcal{X})$ denote, respectively, the algebra of bounded linear operators, the ideal of compact operators, and the set of finite rank operators acting on an infinite dimensional Banach space $\mathcal{X}$. If $T \in \mathcal{B}(\mathcal{X})$ we shall write $N(T)$ and $R(T)$ for the null space and range of T. Also, let $\alpha(T) := \dim\ N(T)$, $\beta(T) := \dim\ \mathcal{X}/R(T)$, and let $\sigma(T)$, $\sigma_a(T)$, $\sigma_p(T)$, $\sigma_{pi}(T)$, $p_0(T)$ and $\pi_0(T)$ denote the spectrum, approximate point spectrum, point spectrum, the eigenvalues of infinite multiplicity of T, the set of poles of T, and the set of all eigenvalues of T which are isolated in $\sigma(T)$, respectively. An operator $T \in \mathcal{B}(\mathcal{X})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space, and is called *lower semi-Fredholm* if it has closed range and its range has finite codimension. If $T \in \mathcal{B}(\mathcal{X})$ is either upper or lower semi-Fredholm, then T is called *semi-Fredholm*; the *index* of a semi Fredholm operator $T \in \mathcal{B}(\mathcal{X})$ is defined as

$$i(T) := \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called *Fredholm*. $T \in \mathcal{B}(\mathcal{X})$ is called *Weyl* if it is Fredholm of index zero, and *Browder* if it is Fredholm "of finite ascent and descent;" equivalently, ([17, Theorem 7.9.3]) if T is Fredholm and $T - \lambda$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum, $\sigma_e(T)$, the Weyl spectrum, $\omega(T)$, and the Browder spectrum, $\sigma_b(T)$, are defined as ([16],[17])

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$
$$\omega(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

and

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc}\ \sigma(T),$$

where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$. For $T \in \mathcal{B}(\mathcal{X})$ and a nonnegative integer n we define $T_{[n]}$ to be the restriction of T to $R(T^n)$, viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range $R(T^n)$ is closed and $T_{[n]}$ is upper (resp. lower) semi-Fredholm, then T is called *upper* (resp. *lower*) *semi-B-Fredholm.* Moreover, if $T_{[n]}$ is Fredholm, then T is called B-Fredholm. T is called *semi-B-Fredholm* if it is upper or lower semi-B-Fredholm.

Definition 1.1. *Let $T \in \mathcal{B}(\mathcal{X})$ and let*

$$\Delta(T) := \{n \in \mathbb{Z}_+ : m \in \mathbb{Z}_+, m \geq n \Rightarrow R(T^n) \cap N(T) \subseteq R(T^m) \cap N(T)\}.$$

The degree of stable iteration of T is defined as dis $T := \inf\ \Delta(T)$.

Let T be semi-B-Fredholm and let d be the degree of stable iteration of T. It follows from [11, Proposition 2.1] that $T_{[m]}$ is semi-Fredholm and $i(T_{[m]}) = i(T_{[d]})$ for every $m \geq d$. This enables us to define the *index* of a *semi-B-Fredholm* operator T as the index of the semi-Fredholm operator $T_{[d]}$. Let $BF(\mathcal{X})$ be the class of all B-Fredholm operators. In [5] the author studied this class of operators and proved [5, Theorem 2.7] that $T \in \mathcal{B}(\mathcal{X})$ is B-Fredholm if and only if $T = T_1 \oplus T_2$, where T_1 is Fredholm and T_2 is nilpotent.

An operator $T \in \mathcal{B}(\mathcal{X})$ is called B-Weyl if it is B-Fredholm of index 0. The B-Fredholm spectrum, $\sigma_{BF}(T)$, and B-Weyl spectrum, $\sigma_{BW}(T)$, are defined as

$$\sigma_{BF}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Fredholm}\}$$

and

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\} \subseteq \sigma_D(T).$$

It is well known that the following equality holds [6]:

$$\sigma_{BW}(T) = \bigcap \{\sigma_D(T + F) : F \in \mathcal{B}_{00}(\mathcal{X})\}.$$

If we write iso $K = K \setminus \text{acc } K$ then we let

$$\pi_{00}(T) := \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$$

and

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T).$$

Given $T \in \mathcal{B}(\mathcal{X})$, we say that Weyl's theorem holds for T (or that T satisfies Weyl's theorem, in symbols, $T \in \mathcal{W}$) if

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T), \tag{1.1}$$

and that Browder's theorem holds for T (in symbols, $T \in \mathcal{B}$) if

$$\sigma(T) \setminus \omega(T) = p_{00}(T). \tag{1.2}$$

We also say that the generalized Weyl's theorem holds for T (and we write $T \in g\mathcal{W}$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T), \tag{1.3}$$

and that the generalized Browder's theorem holds for T (in symbols, $T \in g\mathcal{B}$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = p_0(T). \tag{1.4}$$

It is known ([18],[9]) that

$$g\mathcal{W} \subseteq g\mathcal{B} \bigcap \mathcal{W} \tag{1.5}$$

and that

$$g\mathcal{B} \bigcup \mathcal{W} \subseteq \mathcal{B}. \tag{1.6}$$

Moreover, given $T \in g\mathcal{B}$, it is clear that $T \in g\mathcal{W}$ if and only if $p_0(T) = \pi_0(T)$.

An operator $T \in \mathcal{B}(\mathcal{X})$ is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T. If $T \in \mathcal{B}(\mathcal{X})$, we write $r(T)$ for the spectral radius of T; it is well known that $r(T) \leq ||T||$. An operator $T \in \mathcal{B}(\mathcal{X})$ is called *normaloid* if $r(T) = ||T||$. An operator $X \in \mathcal{B}(\mathcal{X})$ is called a quasiaffinity if it has trivial kernel and dense range. An operator $S \in \mathcal{B}(\mathcal{X})$ is said to be a quasiaffine transform of $T \in \mathcal{B}(\mathcal{X})$ (in symbols, $S \prec T$) if there is a quasiaffinity $X \in \mathcal{B}(\mathcal{X})$ such that $XS = TX$. If both $S \prec T$ and $T \prec S$, then we say that S and T are quasisimilar.

We say that $T \in \mathcal{B}(\mathcal{X})$ has the *single valued extension property* (SVEP) at λ_0 if for every open set $U \subseteq \mathbb{C}$ containing λ_0 the only analytic solution $f : U \longrightarrow \mathcal{X}$ of the equation

$$(T - \lambda)f(\lambda) = 0 \quad (\lambda \in U)$$

is the zero function ([15],[20]). An operator T is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$. Given $T \in \mathcal{B}(\mathcal{X})$, the *local resolvent set* $\rho_T(x)$ of T at the point $x \in \mathcal{X}$ is defined as the union of all open subsets $U \subseteq \mathbb{C}$ for which there is an analytic function $f : U \longrightarrow \mathcal{X}$ such that

$$(T - \lambda)f(\lambda) = x \qquad (\lambda \in U).$$

The *local spectrum* $\sigma_T(x)$ of T at x is then defined as

$$\sigma_T(x) := \mathbb{C} \setminus \rho_T(x).$$

For $T \in \mathcal{B}(\mathcal{X})$, we define the *local* (resp. *glocal*) *spectral subspaces* of T as follows. Given a set $F \subseteq \mathbb{C}$ (resp. a closed set $G \subseteq \mathbb{C}$),

$$X_T(F) := \{x \in \mathcal{X} : \sigma_T(x) \subseteq F\}$$

(resp.

$$\begin{aligned}\mathcal{X}_T(G) := \{x \in \mathcal{X} : &\text{there exists an analytic function}\\ &f : \mathbb{C}\backslash G \to \mathcal{X} \text{ such that } (T-\lambda)f(\lambda) = x \text{ for all } \lambda \in \mathbb{C} \setminus G\}).\end{aligned}$$

An operator $T \in \mathcal{B}(\mathcal{X})$ has *Dunford's property* (C) if the local spectral subspace $X_T(F)$ is closed for every closed set $F \subseteq \mathbb{C}$. We also say that T has *Bishop's property* (β) if for every sequence $f_n : U \to \mathcal{X}$ such that $(T-\lambda)f_n \to 0$ uniformly on compact subsets in U, it follows that $f_n \to 0$ uniformly on compact subsets in U. It is well known [19, 20] that

$$\text{Bishop's property } (\beta) \Longrightarrow \text{Dunford's property } (C) \Longrightarrow \text{SVEP}.$$

2. Structural Properties of Operators in $g\mathcal{B}$ and $g\mathcal{W}$

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{X})$. Then the following statements are equivalent:*
(i) $T \in g\mathcal{B}$;
(ii) $\sigma_{BW}(T) = \sigma_D(T)$;
(iii) $\sigma(T) = \sigma_{BW}(T) \cup \pi_0(T)$;
(iv) $\operatorname{acc} \sigma(T) \subseteq \sigma_{BW}(T)$;
(v) $\sigma(T) \setminus \sigma_{BW}(T) \subseteq \pi_0(T)$.

Proof. (i) $\Longrightarrow$ (ii): Suppose that $T \in g\mathcal{B}$. Then $\sigma(T) \setminus \sigma_{BW}(T) = p_0(T)$. Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $\lambda \in p_0(T)$, and so $T-\lambda$ is Drazin invertible. Therefore $\lambda \in \sigma(T) \setminus \sigma_D(T)$, and hence $\sigma_D(T) \subseteq \sigma_{BW}(T)$. On the other hand, since $\sigma_{BW}(T) \subseteq \sigma_D(T)$ is always true for any operator T, $\sigma_{BW}(T) = \sigma_D(T)$.

(ii) $\Longrightarrow$ (i): We assume that $\sigma_{BW}(T) = \sigma_D(T)$ and we will establish that $\sigma(T) \setminus \sigma_{BW}(T) = p_0(T)$. Suppose first that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_D(T)$, and so $T-\lambda$ is Drazin invertible. Therefore $T-\lambda$ has finite ascent and descent. Since $\lambda \in \sigma(T)$, we have $\lambda \in p_0(T)$. Thus $\sigma(T) \setminus \sigma_{BW}(T) \subseteq p_0(T)$.

Conversely, suppose that $\lambda \in p_0(T)$. Then $T-\lambda$ is Drazin invertible but not invertible. Since λ is an isolated point of $\sigma(T)$, [6, Theorem 4.2] implies that $T-\lambda$ is B-Weyl. Therefore $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Thus $p_0(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$.

(ii) $\Longrightarrow$ (iii): Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_D(T)$, and so $T-\lambda$ is Drazin invertible but not invertible. Therefore $\lambda \in \pi_0(T)$. Thus $\sigma(T) \subseteq \sigma_{BW}(T) \cup \pi_0(T)$. Since $\sigma_{BW}(T) \cup \pi_0(T) \subseteq \sigma(T)$, always, we must have $\sigma(T) = \sigma_{BW}(T) \cup \pi_0(T)$.

(iii) $\Longrightarrow$ (ii): Suppose that $\sigma(T) = \sigma_{BW}(T) \cup \pi_0(T)$. To show that $\sigma_{BW}(T) = \sigma_D(T)$ it suffices to show that $\sigma_D(T) \subseteq \sigma_{BW}(T)$. Suppose that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T-\lambda$ is B-Weyl but not invertible. Since $\sigma(T) = \sigma_{BW}(T) \cup \pi_0(T)$, we see that $\lambda \in \pi_0(T)$. In particular, λ is an isolated point of $\sigma(T)$. It follows from [6, Theorem 4.2] that $T-\lambda$ is Drazin invertible, and hence $\sigma_{BW}(T) = \sigma_D(T)$.

(i) $\Longleftrightarrow$ (iv): Suppose that $T \in g\mathcal{B}$. Then $\sigma(T) \setminus \sigma_{BW}(T) = p_0(T)$. Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $\lambda \in p_0(T)$, and so λ is an isolated point of $\sigma(T)$. Therefore $\lambda \in \sigma(T) \setminus \operatorname{acc} \sigma(T)$, and hence $\operatorname{acc} \sigma(T) \subseteq \sigma_{BW}(T)$.

Conversely, let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since $\operatorname{acc} \sigma(T) \subseteq \sigma_{BW}(T)$, it follows that $\lambda \in \operatorname{iso} \sigma(T)$ and $T-\lambda$ is B-Weyl. By [7, Theorem 2.3], we must have $\lambda \in p_0(T)$. Therefore $\sigma(T) \setminus \sigma_{BW}(T) \subseteq p_0(T)$. For the converse, suppose that $\lambda \in p_0(T)$. Then λ is a pole of the resolvent of T, and so λ is an isolated point of $\sigma(T)$. Therefore $\lambda \in \sigma(T) \setminus \text{acc } \sigma(T)$. It follows from [7, Theorem 2.3] that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Thus $p_0(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$, and so $T \in g\mathcal{B}$.

(iv) $\Longleftrightarrow$ (v): Suppose that $\text{acc } \sigma(T) \subseteq \sigma_{BW}(T)$, and let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda$ is B-Weyl but not invertible. Since $\text{acc } \sigma(T) \subseteq \sigma_{BW}(T)$, λ is an isolated point of $\sigma(T)$. It follows from [7, Theorem 2.3] that λ is a pole of the resolvent of T. Therefore $\lambda \in \pi_0(T)$, and hence $\sigma(T)\setminus\sigma_{BW}(T) \subseteq \pi_0(T)$. Conversely, suppose that $\sigma(T)\setminus\sigma_{BW}(T) \subseteq \pi_0(T)$ and let $\lambda \in \sigma(T)\setminus\sigma_{BW}(T)$. Then $\lambda \in \pi_0(T)$, and so λ is an isolated point of $\sigma(T)$. Therefore $\lambda \in \sigma(T) \setminus \text{acc } \sigma(T)$, which implies that $\text{acc } \sigma(T) \subseteq \sigma_{BW}(T)$. $\square$

Corollary 2.2. *Let T be quasinilpotent or algebraic. Then $T \in g\mathcal{B}$.*

Proof. Straightforward from Theorem 2.1 and the fact that $\text{acc } \sigma(T) = \emptyset$ whenever T is quasinilpotent or algebraic. $\square$

Recall that $g\mathcal{W} \subseteq g\mathcal{B}$ (cf. (1.5)). However, the reverse inclusion does not hold, as the following example shows.

Example 2.3. Let $\mathcal{X} = \ell_p$, let $T_1, T_2 \in \mathcal{B}(\mathcal{X})$ be given by

$$T_1(x_1, x_2, x_3, \cdots) := (0, \frac{1}{2}x_1, \frac{1}{3}x_2, \frac{1}{4}x_3, \cdots) \text{ and } T_2 := 0,$$

and let

$$T := \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \in \mathcal{B}(\mathcal{X} \oplus \mathcal{X}).$$

Then

$$\sigma(T) = \omega(T) = \sigma_{BW}(T) = \pi_0(T) = \{0\}$$

and

$$p_0(T) = \emptyset.$$

Therefore, $T \in g\mathcal{B} \setminus g\mathcal{W}$.

The next result gives simple necessary and sufficient conditions for an operator $T \in g\mathcal{B}$ to belong to the smaller class $g\mathcal{W}$.

Theorem 2.4. *Let $T \in g\mathcal{B}$. The following statements are equivalent.*
(i) $T \in g\mathcal{W}$.
(ii) $\sigma_{BW}(T) \cap \pi_0(T) = \emptyset$.
(iii) $p_0(T) = \pi_0(T)$.

Proof. (i) $\Rightarrow$ (ii): Assume $T \in g\mathcal{W}$, that is, $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$. It then follows easily that $\sigma_{BW}(T) \cap \pi_0(T) = \emptyset$, as required for (ii).

(ii) $\Rightarrow$ (iii): Let $\lambda \in \pi_0(T)$. The condition in (ii) implies that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, and since $T \in g\mathcal{B}$, we must then have $\lambda \in p_0(T)$. It follows that $\pi_0(T) \subseteq p_0(T)$, and since the reverse inclusion always hold, we obtain (iii).

(iii) $\Rightarrow$ (i): Since $T \in g\mathcal{B}$, we know that $\sigma(T) \setminus \sigma_{BW}(T) = p_0(T)$, and since we are assuming $p_0(T) = \pi_0(T)$, it follows that $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$, that is, $T \in g\mathcal{W}$. $\square$

It is well known that $\sigma_b(T) = \sigma_e(T) \cup \text{acc } \sigma(T)$. A similar result holds for the Drazin spectrum.

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{X})$. Then $\sigma_D(T) = \sigma_{BF}(T) \cup \text{acc } \sigma(T)$.*

Proof. Suppose that $\lambda \in \sigma(T) \setminus \sigma_D(T)$. Then $T - \lambda$ is Drazin invertible but not invertible. Therefore $T - \lambda$ has finite ascent and descent, and hence $T - \lambda$ can be decomposed as $T - \lambda = T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent. It follows from [6, Lemma 4.1] that $T - \lambda$ is B-Fredholm. On the other hand, since $T - \lambda$ has finite ascent and descent, λ is an isolated point of $\sigma(T)$. Hence $\lambda \in \sigma(T) \setminus (\sigma_{BF}(T) \cup \mathrm{acc}\ \sigma(T))$.

Conversely, suppose that $\lambda \in \sigma(T) \setminus (\sigma_{BF}(T) \cup \mathrm{acc}\ \sigma(T))$. Then $T - \lambda$ is B-Fredholm and λ is an isolated point of $\sigma(T)$. Since $T - \lambda$ is B-Fredholm, it follows from [5, Theorem 2.7] that $T - \lambda$ can be decomposed as $T - \lambda = T_1 \oplus T_2$, where T_1 is Fredholm and T_2 is nilpotent. We consider two cases.

Case I. Suppose that T_1 is invertible. Then $T - \lambda$ is Drazin invertible, and so $\lambda \notin \sigma_D(T)$.

Case II. Suppose that T_1 is not invertible. Then 0 is an isolated point of $\sigma(T_1)$. But T_1 is a Fredholm operator, hence it follows from the punctured neighborhood theorem that T_1 is Browder. Therefore there exists a finite rank operator S_1 such that $T_1 + S_1$ is invertible and $T_1 S_1 = S_1 T_1$. Put $F := S_1 \oplus 0$. Then F is a finite rank operator, $TF = FT$ and

$$T - \lambda + F = T_1 \oplus T_2 + S_1 \oplus 0 = (T_1 + S_1) \oplus T_2$$

is Drazin invertible. Hence $\lambda \notin \sigma_D(T)$. □

In general, the spectral mapping theorem does not hold for the B-Weyl spectrum, as shown by the following example.

Example 2.6. Let $U \in B(l_2)$ be the unilateral shift and consider the operator

$$T := U \oplus (U^* + 2).$$

Let $p(z) := z(z - 2)$. Since U is Fredholm with $i(U) = -1$ and since $U - 2$ and $U^* + 2$ are both invertible, it follows that T and $T - 2$ are Fredholm with indices -1 and 1, respectively. Therefore T and $T - 2$ are both B-Fredholm but T is not B-Weyl. On the other hand, it follows from the index product theorem that

$$i(p(T)) = i(T(T - 2)) = i(T) + i(T - 2) = 0,$$

hence $p(T)$ is Weyl. Thus $0 \notin \sigma_{BW}(p(T))$, whereas $0 = p(0) \in p(\sigma_{BW}(T))$.

M. Berkani and M. Sarih have shown in [10] that the spectral mapping theorem holds for the Drazin spectrum. We give here an alternative proof using Theorem 2.5.

Theorem 2.7. *Let $T \in \mathcal{B}(\mathcal{X})$ and let $f \in H(\sigma(T))$. Then*

$$\sigma_D(f(T)) = f(\sigma_D(T)).$$

Proof. Suppose that $\mu \notin f(\sigma_D(T))$ and set

$$h(\lambda) := f(\lambda) - \mu.$$

Then h has no zeros in $\sigma_D(T)$. Since $\sigma_D(T) = \sigma_{BF}(T) \cup \mathrm{acc}\ \sigma(T)$ by Theorem 2.5, we conclude that h has finitely many zeros in $\sigma(T)$. Now we consider two cases.

Case I. Suppose that h has no zeros in $\sigma(T)$. Then $h(T) = f(T) - \mu$ is invertible, and so $\mu \notin \sigma_D(f(T))$.

Case II. Suppose that h has at least one zero in $\sigma(T)$. Then

$$h(\lambda) \equiv c_0(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) g(\lambda),$$

where $c_0, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ and $g(\lambda)$ is a nonvanishing analytic function on an open neighborhood. Therefore

$$h(T) = c_0(T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_n) g(T),$$

where $g(T)$ is invertible. Since $\mu \notin f(\sigma_D(T))$, $\lambda_1, \lambda_2, \ldots, \lambda_n \notin \sigma_D(T)$. Therefore $T - \lambda_i$ is Drazin invertible, and hence each $T-\lambda_i$ is B-Weyl ($i = 1, 2, \ldots, n$). But each λ_i is an isolated point of $\sigma(T)$, hence it follows from [7, Theorem 2.3] that each λ_i is a pole of the resolvent of T. Therefore $T - \lambda_i$ has finite ascent and descent ($i = 1, 2, \ldots, n$), so $(T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n)$ has finite ascent and descent by [23, Theorem 7.1]. Since $g(T)$ is invertible, $h(T)$ has finite ascent and descent. Therefore $h(T)$ is Drazin invertible, and so $0 \notin \sigma_D(h(T))$. Hence $\mu \notin \sigma_D(f(T))$. It follows from Cases I and II that $\sigma_D(f(T)) \subseteq f(\sigma_D(T))$.

Conversely, suppose that $\lambda \notin \sigma_D(f(T))$. Then $f(T) - \lambda$ is Drazin invertible. We again consider two cases.

Case I. Suppose that $f(T) - \lambda$ is invertible. Then $\lambda \notin \sigma(f(T)) = f(\sigma(T))$, and hence $\lambda \notin f(\sigma_D(T))$.

Case II. Suppose that $\lambda \in \sigma(f(T)) \setminus \sigma_D(f(T))$. Write

$$f(T) - \lambda \equiv c_0(T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n)g(T),$$

where $c_0, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ and $g(T)$ is invertible. Since $f(T) - \lambda$ is Drazin invertible, $f(T) - \lambda = c_0(T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n)g(T)$ has finite ascent and descent. Hence $T - \lambda_i$ has finite ascent and descent for every $i = 1, 2, \ldots, n$ by [23, Theorem 7.1]. Therefore each $T - \lambda_i$ is Drazin invertible, and so $\lambda_1, \lambda_2, \ldots, \lambda_n \notin \sigma_D(T)$.

We now wish to prove that $\lambda \notin f(\sigma_D(T))$. Assume not; then there exists a $\mu \in \sigma_D(T)$ such that $f(\mu) = \lambda$. Since $g(\mu) \neq 0$, we must have $\mu = \mu_i$ for some $i = 1, ..., n$, which implies $\mu_i \in \sigma_D(T)$, a contradiction. Hence $\lambda \notin f(\sigma_D(T))$, and so $f(\sigma_D(T)) \subseteq \sigma_D(f(T))$. This completes the proof. $\square$

Let $T \in \mathcal{B}(\mathcal{X})$ and let $f \in H(\sigma(T))$, where $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$. It is well known that $\omega(f(T)) \subseteq f(\omega(T))$ holds. The following corollary shows that a similar result holds for the B-Weyl spectrum with some additional condition.

Corollary 2.8. *Let $T \in g\mathcal{B}$ and let $f \in H(\sigma(T))$. Then*

$$\sigma_{BW}(f(T)) \subseteq f(\sigma_{BW}(T)). \tag{2.1}$$

Proof. Since $T \in g\mathcal{B}$, it follows from Theorem 2.1 that $\sigma_{BW}(T) = \sigma_D(T)$. By Theorem 2.7 we have

$$\sigma_{BW}(f(T)) \subseteq \sigma_D(f(T)) = f(\sigma_D(T)) = f(\sigma_{BW}(T)).$$

Thus $\sigma_{BW}(f(T)) \subseteq f(\sigma_{BW}(T))$. $\square$

We obtain the following theorem, which extends a result in [13].

Theorem 2.9. *Let $S, T \in \mathcal{B}(\mathcal{X})$. If T has SVEP and $S \prec T$, then $f(S) \in g\mathcal{B}$ for every $f \in H(\sigma(S))$. In particular, if T has SVEP then $T \in g\mathcal{B}$.*

Proof. Suppose that T has SVEP. Since $S \prec T$, it follows from the proof of [13, Theorem 3.2] that S has SVEP. We now show that $S \in g\mathcal{B}$. Let $\lambda \in \sigma(S) \setminus \sigma_{BW}(S)$; then $S - \lambda$ is B-Weyl but not invertible. Since $S-\lambda$ is B-Weyl, it follows from [6, Lemma 4.1] that $S-\lambda$ admits the decomposition $S - \lambda = S_1 \oplus S_2$,where S_1 is Weyl and S_2 is nilpotent. Since S has SVEP, S_1 and S_2 also have SVEP. Therefore Browder's theorem holds for S_1, and hence $\omega(S_1) = \sigma_b(S_1)$. Since S_1 is Weyl, S_1 is Browder. Hence λ is an isolated point of $\sigma(S)$. It follows from Theorem 2.1 that $S \in g\mathcal{B}$.

Now let $f \in H(\sigma(S))$. Since S has SVEP, it follows from [20, Theorem 3.3.6] that $f(S)$ has SVEP. Therefore $f(S) \in g\mathcal{B}$, by the first part of the proof. $\square$

We now recall that the generalized Weyl's theorem may not hold for quasinilpotent operators, and that it does not necessarily transfer to or from adjoints.

Example 2.10. On $\mathcal{X} \equiv \ell_p$ let

$$T(x_1, x_2, x_3, \cdots) := (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots).$$

Then

$$\sigma(T^*) = \sigma_{BW}(T^*) = \{0\}$$

and

$$\pi_0(T^*) = \emptyset.$$

Therefore $T^* \in g\mathcal{W}$. On the other hand, since $\sigma(T) = \omega(T) = \pi_{00}(T)$, $T \notin \mathcal{W}$. Hence $T \notin g\mathcal{W}$.

However, the generalized Browder's theorem performs better.

Theorem 2.11. *Let $T \in \mathcal{B}(\mathcal{X})$. Then the following statements are equivalent:*
(i) $T \in g\mathcal{B}$;
(ii) $T^* \in g\mathcal{B}$.

Proof. Recall that

$$\sigma(T) = \sigma(T^*) \text{ and } \sigma_{BW}(T) = \sigma_{BW}(T^*).$$

Therefore,

$$\text{acc } \sigma(T) \subseteq \sigma_{BW}(T) \Longleftrightarrow \text{acc } \sigma(T^*) \subseteq \sigma_{BW}(T^*).$$

It follows from Theorem 2.1 that $T \in g\mathcal{B}$ if and only if $T^* \in g\mathcal{B}$. $\square$

3. Operators Reduced by Their Eigenspaces

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and suppose that $T \in \mathcal{B}(\mathcal{H})$ is reduced by each of its eigenspaces. If we let

$$\mathfrak{M} := \bigvee \{N(T - \lambda) : \ \lambda \in \sigma_p(T)\},$$

it follows that $\mathfrak{M}$ reduces T. Let $T_1 := T|\mathfrak{M}$ and $T_2 := T|\mathfrak{M}^{\perp}$. By [4, Proposition 4.1] we have:

(i) T_1 is a normal operator with pure point spectrum;
(ii) $\sigma_p(T_1) = \sigma_p(T)$;
(iii) $\sigma(T_1) = \text{cl}\, \sigma_p(T_1)$ (here cl denotes closure);
(iv) $\sigma_p(T_2) = \emptyset$.

In [4, Definition 5.4], Berberian defined

$$\tau(T) := \sigma(T_2) \cup \text{acc } \sigma_p(T) \cup \sigma_{pi}(T);$$

we shall call $\tau(T)$ the *Berberian spectrum* of T. Berberian proved that $\tau(T)$ is a nonempty compact subset of $\sigma(T)$. In the following theorem we establish a relation amongst the B-Weyl, the Drazin and the Berberian spectra.

Theorem 3.1. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is reduced by each of its eigenspaces. Then*

$$\sigma_{BW}(T) = \sigma_D(T) \subseteq \tau(T). \tag{3.1}$$

Proof. Let $\mathfrak{M}$ be the closed linear span of the eigenspaces $N(T - \lambda)$ ($\lambda \in \sigma_p(T)$) and write

$$T_1 := T|\mathfrak{M} \text{ and } T_2 := T|\mathfrak{M}^{\perp}.$$

From the preceding arguments it follows that T_1 is normal, $\sigma_p(T_1) = \sigma_p(T)$ and $\sigma_p(T_2) = \emptyset$. Toward (3.1) we will show that

$$\sigma_{BW}(T) \subseteq \tau(T) \tag{3.2}$$

and

$$\sigma_D(T) \subseteq \sigma_{BW}(T). \tag{3.3}$$

To establish (3.2) suppose that $\lambda \in \sigma(T) \setminus \tau(T)$. Then $T_2 - \lambda$ is invertible and $\lambda \in \pi_0(T_1)$. Since $\sigma_{pi}(T) \subseteq \tau(T)$, we see that $\lambda \in \pi_{00}(T_1)$. Since T_1 is normal, it follows from [6, Theorem 4.5] that $T_1 \in g\mathcal{W}$. Therefore $\lambda \in \sigma(T_1) \setminus \sigma_{BW}(T_1)$, and hence $T - \lambda$ is B-Weyl. This proves (3.2).

Toward (3.3) suppose that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda$ is B-Weyl but not invertible. Observe that if $\mathcal{H}_1$ is a Hilbert space and an operator $R \in \mathcal{B}(\mathcal{H}_1)$ satisfies $\sigma_{BW}(R) = \sigma_{BF}(R)$, then

$$\sigma_{BW}(R \oplus S) = \sigma_{BW}(R) \cup \sigma_{BW}(S), \tag{3.4}$$

for every Hilbert space $\mathcal{H}_2$ and $S \in \mathcal{B}(\mathcal{H}_2)$. Indeed, if $\lambda \notin \sigma_{BW}(R) \cup \sigma_{BW}(S)$, then $R - \lambda$ and $S - \lambda$ are both B-Weyl. Therefore $R - \lambda$ and $S - \lambda$ are B-Fredholm with index 0. Hence $R - \lambda \oplus S - \lambda$ is B-Fredholm; moreover,

$$i((R - \lambda) \oplus (S - \lambda)) = i(R - \lambda) + i(S - \lambda) = 0.$$

Therefore $R \oplus S - \lambda$ is B-Weyl, and so $\lambda \notin \sigma_{BW}(R \oplus S)$, which implies $\sigma_{BW}(R \oplus S) \subseteq \sigma_{BW}(R) \cup \sigma_{BW}(S)$. Conversely, suppose that $\lambda \notin \sigma_{BW}(R \oplus S)$. Then $R \oplus S - \lambda$ is B-Fredholm with index 0. Since $i(R \oplus S - \lambda) = i(R - \lambda) + i(S - \lambda)$ and $i(R - \lambda) = 0$, we must have $i(S - \lambda) = 0$. Therefore $R - \lambda$ and $S - \lambda$ are both B-Weyl. Hence $\lambda \notin \sigma_{BW}(R) \cup \sigma_{BW}(S)$, which implies $\sigma_{BW}(R) \cup \sigma_{BW}(S) \subseteq \sigma_{BW}(R \oplus S)$. Since T_1 is normal, we can now apply (3.4) to T_1 in place of R to show that $T_1 - \lambda$ and $T_2 - \lambda$ are both B-Weyl. But since $\sigma_p(T_2) = \emptyset$, we see that $T_2 - \lambda$ is Weyl and injective. Therefore $T_2 - \lambda$ is invertible, and so $\lambda \in \sigma(T_1) \setminus \sigma_{BW}(T_1)$. Since T_1 is normal, it follows from [6, Theorem 4.5] that $T_1 \in g\mathcal{W}$, which implies $\lambda \in \pi_0(T_1)$. Hence λ is an isolated point of $\sigma(T_1)$ and $T_2 - \lambda$ is invertible. Now observe that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces then the following equality holds with no other restriction on either R or S:

$$\sigma_D(R \oplus S) = \sigma_D(R) \cup \sigma_D(S), \tag{3.5}$$

for every $R \in B(\mathcal{H}_1)$ and $S \in B(\mathcal{H}_2)$. Indeed, if $\lambda \notin \sigma_D(R) \cup \sigma_D(S)$, then $R - \lambda$ and $S - \lambda$ are both Drazin invertible. It follows that each of $R - \lambda$ and $S - \lambda$ can be written as the direct sum of an invertible operator and a nilpotent operator, and the same is therefore true of the direct sum $(R - \lambda) \oplus (S - \lambda) \equiv R \oplus S - \lambda$. Thus, $\lambda \notin \sigma_D(R \oplus S)$, and hence $\sigma_D(R \oplus S) \subseteq \sigma_D(R) \cup \sigma_D(S)$. Conversely, suppose that $\lambda \notin \sigma_D(R \oplus S)$. It follows from Theorem 2.5 that $(R - \lambda) \oplus (S - \lambda)$ is B-Fredholm and λ is an isolated point of $\sigma(R \oplus S)$. Since $\sigma(R \oplus S) = \sigma(R) \cup \sigma(S)$, it follows that $R - \lambda$ and $S - \lambda$ are both B-Fredholm, and λ is an isolated point of $\sigma(R)$ and $\sigma(S)$, respectively. It follows from Theorem 2.5 that $R - \lambda$ and $S - \lambda$ are both Drazin invertible. Therefore $\lambda \notin \sigma_D(R) \cup \sigma_D(S)$, and hence $\sigma_D(R) \cup \sigma_D(S) \subseteq \sigma_D(R \oplus S)$. We have thus established (3.5).

Now, by Theorem 2.5 and (3.5) we have $\lambda \notin \sigma_D(T)$. This proves (3.3) and completes the proof of the Theorem. □

In [21], Oberai showed that if $T \in \mathcal{B}(\mathcal{X})$ is isoloid and if $T \in \mathcal{W}$ then for any polynomial p, $p(T) \in \mathcal{W}$ if and only if $\omega(p(T)) = p(\omega(T))$. We now show that a similar result holds for the generalized Weyl's theorem. We begin with the following two lemmas, essentially due to Oberai [21]; we include proofs for the reader's convenience.

Lemma 3.2. *Let $T \in \mathcal{B}(\mathcal{X})$ and let $f \in H(\sigma(T))$. Then*

$$\sigma(f(T)) \setminus \pi_0(f(T)) \subseteq f(\sigma(T) \setminus \pi_0(T)).$$

Proof. Suppose that $\lambda \in \sigma(f(T)) \setminus \pi_0(f(T))$. By the spectral mapping theorem, it follows that $\lambda \in f(\sigma(T)) \setminus \pi_0(f(T))$. We consider two cases.

Case I. Suppose that λ is not an isolated point of $f(\sigma(T))$. Then there exists a sequence $\{\lambda_n\} \subseteq f(\sigma(T))$ such that $\lambda_n \to \lambda$. Since $\lambda_n \in f(\sigma(T))$, $\lambda_n = f(\mu_n)$ for some $\mu_n \in \sigma(T)$. By the compactness of $\sigma(T)$, there is a convergent subsequence $\{\mu_{n_k}\}$ such that $\mu_{n_k} \to \mu \in \sigma(T)$. It follows that $f(\mu_{n_k}) \to \lambda$, and therefore $\lambda = f(\mu)$. But $\mu \in \sigma(T) \setminus \pi_0(T)$, whence $\lambda = f(\mu) \in f(\sigma(T) \setminus \pi_0(T))$.

Case II. Suppose now that λ is an isolated point of $f(\sigma(T))$. Since $\lambda \in \pi_0(f(T))$ by assumption, it follows that λ cannot be an eigenvalue of $f(T)$. Let

$$f(T) - \lambda = c_0(T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_n)g(T), \tag{3.6}$$

where $c_0, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $g(T)$ is invertible. Since $f(T) - \lambda$ is injective, and the operators on the right-hand side of (3.6) commute, none of $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be an eigenvalue of T. Therefore $\lambda \in f(\sigma(T) \setminus \pi_0(T))$.

From Cases I and II we obtain the desired conclusion. $\square$

Lemma 3.3. *Let $T \in \mathcal{B}(\mathcal{X})$ and assume that T is isoloid. Then for any $f \in H(\sigma(T))$ we have*

$$\sigma(f(T)) \setminus \pi_0(f(T)) = f(\sigma(T) \setminus \pi_0(T)).$$

Proof. In view of Lemma 3.2 it suffices to prove that $f(\sigma(T) \setminus \pi_0(T)) \subseteq \sigma(f(T)) \setminus \pi_0(f(T))$. Suppose that $\lambda \in f(\sigma(T) \setminus \pi_0(T))$. Then by the spectral mapping theorem, we must have $\lambda \in \sigma(f(T))$. Assume that $\lambda \in \pi_0(f(T))$. Then clearly, λ is an isolated point of $\sigma(f(T))$. Let

$$f(T) - \lambda = c_0(T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_n)g(T),$$

where $c_0, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $g(T)$ is invertible. If for some $i = 1, ..., n$, $\lambda_i \in \sigma(T)$, then λ_i would be an isolated point of $\sigma(T)$. But T is isoloid, hence λ_i would also be an eigenvalue of T. Since $\lambda \in \pi_0(f(T))$, such λ_i would belong to $\pi_0(T)$. Thus, $\lambda = f(\lambda_i)$ for some $\lambda_i \in \pi_0(T)$, and hence $\lambda \in f(\pi_0(T))$, a contradiction. Therefore $\lambda \notin \pi_0(f(T))$, so that $\lambda \in \sigma(f(T)) \setminus \pi_0(f(T))$. $\square$

Theorem 3.4. *Suppose that $T \in \mathcal{B}(\mathcal{X})$ is isoloid and $T \in g\mathcal{W}$. Then for any $f \in H(\sigma(T))$,*

$$f(T) \in g\mathcal{W} \Longleftrightarrow f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)).$$

Proof. ($\Longrightarrow$) Suppose $f(T) \in g\mathcal{W}$. Then $\sigma_{BW}(f(T)) = \sigma(f(T)) \setminus \pi_0(f(T))$. Since T is isoloid, it follows from Lemma 3.3 that $f(\sigma(T) \setminus \pi_0(T)) = \sigma(f(T)) \setminus \pi_0(f(T))$. But $T \in g\mathcal{W}$, hence $\sigma_{BW}(T) = \sigma(T) \setminus \pi_0(T)$, which implies $f(\sigma_{BW}(T)) = f(\sigma(T) \setminus \pi_0(T))$. Therefore

$$\begin{aligned} f(\sigma_{BW}(T)) &= f(\sigma(T) \setminus \pi_0(T)) \\ &= \sigma(f(T)) \setminus \pi_0(f(T)) = \sigma_{BW}(f(T)). \end{aligned}$$

($\Longleftarrow$) Suppose that $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$. Since T is isoloid, it follows from Lemma 3.3 that $f(\sigma(T) \setminus \pi_0(T)) = \sigma(f(T)) \setminus \pi_0(f(T))$. Since $T \in g\mathcal{W}$, we have $\sigma_{BW}(T) = \sigma(T) \setminus \pi_0(T)$. Therefore

$$\begin{aligned} \sigma_{BW}(f(T)) &= f(\sigma_{BW}(T)) \\ &= f(\sigma(T) \setminus \pi_0(T)) = \sigma(f(T)) \setminus \pi_0(f(T)), \end{aligned}$$

and hence $f(T) \in g\mathcal{W}$. $\square$

As applications of Theorems 3.1 and 3.4 we will obtain below several corollaries.

Corollary 3.5. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is reduced by each of its eigenspaces. Then $f(T) \in g\mathcal{B}$ for every $f \in H(\sigma(T))$. In particular, $T \in g\mathcal{B}$.*

Proof. Since T is reduced by each of its eigenspaces, $T - \lambda$ has finite ascent for each $\lambda \in \mathbb{C}$. Therefore T has SVEP, and hence by [20, Theorem 3.3.6] $f(T)$ has SVEP for each $f \in H(\sigma(T))$. It follows from Theorem 2.9 that $f(T) \in g\mathcal{B}$. $\square$

In Example 2.10 we already noticed that the generalized Weyl's theorem does not transfer to or from adjoints. However, we have:

Corollary 3.6. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is reduced by each of its eigenspaces, and assume that $\sigma(T)$ has no isolated points. Then $T, T^* \in g\mathcal{W}$. Moreover, if $f \in H(\sigma(T))$ then $f(T) \in g\mathcal{W}$.*

Proof. We first show that $T \in g\mathcal{W}$. Since T is reduced by each of its eigenspaces, it follows from Theorem 3.4 that $T \in g\mathcal{B}$. By Theorem 2.1, $\sigma(T) \setminus \sigma_{BW}(T) \subseteq \pi_0(T)$. But iso $\sigma(T) = \emptyset$, hence $\pi_0(T) = \emptyset$, which implies $\sigma_{BW}(T) = \sigma(T)$. Therefore, $T \in g\mathcal{W}$. On the other hand, observe that

$$\sigma(T^*) = \overline{\sigma(T)},\ \sigma_{BW}(T^*) = \overline{\sigma_{BW}(T)},$$

and

$$\pi_0(T^*) = \overline{\pi_0(T)} = \emptyset.$$

Hence $T^* \in g\mathcal{W}$. Let $f \in H(\sigma(T))$. Since T is reduced by each of its eigenvalues, T has SVEP. It follows from [20, Theorem 3.3.6] that $f(T)$ has SVEP. Therefore, by Theorems 2.1 and 2.7,

$$\sigma_{BW}(f(T)) = \sigma_D(f(T)) = f(\sigma_D(T)) = f(\sigma_{BW}(T)).$$

Thus $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$. But $\sigma(T)$ has no isolated points, hence T is isoloid. It follows from Theorem 3.4 that generalized Weyl's theorem holds for $f(T)$. □

For the next result, we recall that an operator T is called *reduction-isoloid* if the restriction of T to every reducing subspace is isoloid; it is well known that hyponormal operators are reduction-isoloid [22].

Corollary 3.7. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is both reduction-isoloid and reduced by each of its eigenspaces. Then $f(T) \in g\mathcal{W}$ for every $f \in H(\sigma(T))$.*

Proof. We first show that $T \in g\mathcal{W}$. In view of Theorem 3.4, it suffices to show that $\pi_0(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$. Suppose that $\lambda \in \pi_0(T)$. Then, with the preceding notations,

$$\lambda \in \pi_0(T_1) \cap [\text{iso } \sigma(T_2) \cup \rho(T_2)].$$

If $\lambda \in$ iso $\sigma(T_2)$, then since T_2 is isoloid we have $\lambda \in \sigma_p(T_2)$. But $\sigma_p(T_2) = \emptyset$, hence we must have $\lambda \in \pi_0(T_1) \cap \rho(T_2)$. Since T_1 is normal, $T_1 \in g\mathcal{W}$. Hence $T_1 - \lambda$ is B-Weyl and so is $T - \lambda$, which implies $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Therefore $\pi_0(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$, and hence $T \in g\mathcal{W}$. Now, let $f \in H(\sigma(T))$. Since T is reduced by each of its eigenspaces, it follows from the proof of Corollary 3.6 that $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$. Therefore $f(T) \in g\mathcal{W}$ by Theorem 3.4. □

4. Applications

In [6] and [7], the authors showed that the generalized Weyl's theorem holds for normal operators. In this section we extend this result to algebraically M-hyponormal operators and to algebraically paranormal operators, using the results in Sections 2 and 3. We begin with the following definition.

Definition 4.1. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be M-hyponormal if there exists a positive real number M such that*

$$M||(T-\lambda)x|| \geq ||(T-\lambda)^*x|| \quad \text{for all } x \in \mathcal{H},\ \lambda \in \mathbb{C}.$$

We say that $T \in \mathcal{B}(\mathcal{H})$ is algebraically M-hyponormal if there exists a nonconstant complex polynomial p such that $p(T)$ is M-hyponormal.

The following implications hold:

$$\text{hyponormal} \Longrightarrow M\text{-hyponormal} \Longrightarrow \text{algebraically } M\text{-hyponormal}.$$

The following result follows from Definition 4.1 and some well known facts about M-hyponormal operators.

Lemma 4.2. *(i) If T is algebraically M-hyponormal then so is $T-\lambda$ for every $\lambda \in \mathbb{C}$.*
(ii) If T is algebraically M-hyponormal and $\mathcal{M} \subseteq \mathcal{H}$ is invariant under T, then $T|\mathcal{M}$ is algebraically M-hyponormal.
(iii) If T is M-hyponormal, then $N(T-\lambda) \subseteq N(T-\lambda)^*$ for every $\lambda \in \mathbb{C}$.
(iv) Every quasinilpotent M-hyponormal operator is the zero operator.

In [2], Arora and Kumar proved that Weyl's theorem holds for every M-hyponormal operator. We shall show that the generalized Weyl's theorem holds for algebraically M-hyponormal operators. To do this, we need several preliminary results.

Lemma 4.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be M-hyponormal, let $\lambda \in \mathbb{C}$, and assume that $\sigma(T) = \{\lambda\}$. Then $T = \lambda$.*

Proof. Since T is M-hyponormal, $T-\lambda$ is also M-hyponormal. Since $T-\lambda$ is quasinilpotent, (iv) above implies that $T-\lambda = 0$. $\square$

Lemma 4.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasinilpotent algebraically M-hyponormal operator. Then T is nilpotent.*

Proof. Let p be a nonconstant polynomial such that $p(T)$ is M-hyponormal. Since $\sigma(p(T)) = p(\sigma(T))$, the operator $p(T)-p(0)$ is quasinilpotent. It follows from Lemma 4.3 that $c\, T^m(T-\lambda_1)\cdots(T-\lambda_n) \equiv p(T)-p(0) = 0$. Since $T-\lambda_i$ is invertible for every $\lambda_i \neq 0$, we must have $T^m = 0$. $\square$

It is well known that every M-hyponormal operator is isoloid. We can extend this result to the algebraically M-hyponormal operators.

Lemma 4.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be an algebraically M-hyponormal operator. Then T is isoloid.*

Proof. Let λ be an isolated point of $\sigma(T)$. Using the spectral projection $P := \frac{1}{2\pi i}\int_{\partial B}(\mu - T)^{-1} d\mu$, where B is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum $T = T_1 \oplus T_2$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T)\setminus\{\lambda\}$. Since T is algebraically M-hyponormal, $p(T)$ is M-hyponormal for some nonconstant polynomial p. Since $\sigma(T_1) = \{\lambda\}$, $\sigma(p(T_1)) = p(\sigma(T_1)) = \{p(\lambda)\}$. Therefore $p(T_1)-p(\lambda)$ is quasinilpotent. Since $p(T_1)$ is M-hyponormal, it follows from Lemma 4.3 that $p(T_1)-p(\lambda) = 0$. Put $q(z) := p(z)-p(\lambda)$. Then $q(T_1) = 0$, and hence T_1 is algebraically M-hyponormal. Since $T_1-\lambda$ is quasinilpotent and algebraically M-hyponormal, it follows from Lemma 4.4 that $T_1-\lambda$ is nilpotent. Therefore $\lambda \in \sigma_p(T_1)$, and hence $\lambda \in \sigma_p(T)$. This shows that T is isoloid. $\square$

Lemma 4.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be an algebraically M-hyponormal operator. Then T has finite ascent. In particular, every algebraically M-hyponormal operator has SVEP.*

Proof. Suppose $p(T)$ is M-hyponormal for some nonconstant polynomial p. Since M-hyponormality is translation-invariant, we may assume $p(0) = 0$. If $p(\lambda) \equiv a_0\lambda^m$, then $N(T^m) = N(T^{2m})$ because M-hyponormal operators are of ascent 1. Thus we write $p(\lambda) \equiv a_0\,\lambda^m(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$ ($m \neq 0$; $\lambda_i \neq 0$ for $1 \leq i \leq n$). We then claim that

$$N(T^m) = N(T^{m+1}). \tag{4.1}$$

To show (4.1), let $0 \neq x \in N(T^{m+1})$. Then we can write

$$p(T)x = (-1)^n \, a_0 \, \lambda_1 \cdots \lambda_n \, T^m x.$$

Thus we have

$$\begin{aligned}
|a_0\lambda_1 \cdots \lambda_n|^2 ||T^m x||^2 &= (p(T)x, \; p(T)x) \\
&\leq ||p(T)^* p(T)x|| \, ||x|| \\
&\leq M||p(T)^2 x|| \, ||x|| \quad \text{(because } p(T) \text{ is } M\text{-hyponormal)} \\
&= M||a_0^2 \, (T - \lambda_1 I)^2 \cdots (T - \lambda_n I)^2 T^{2m} x|| \, ||x|| \\
&= 0,
\end{aligned}$$

which implies $x \in N(T^m)$. Therefore $N(T^{m+1}) \subseteq N(T^m)$ and the reverse inclusion is always true. Since every algebraically M-hyponormal operator has finite ascent, it follows from [19, Proposition 1.8] that every algebraically M-hyponormal operator has SVEP. $\square$

Theorem 4.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be an algebraically M-hyponormal operator. Then $f(T) \in g\mathcal{W}$ for every $f \in H(\sigma(T))$.*

Proof. We first show that $T \in g\mathcal{W}$. Suppose that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda$ is B-Weyl but not invertible. Since T is algebraically M-hyponormal, there exists a nonconstant polynomial p such that $p(T)$ is M-hyponormal. Since every algebraically M-hyponormal operator has SVEP by Lemma 4.6, T has SVEP. It follows from Theorem 2.9 that $T \in g\mathcal{B}$. Therefore $\sigma_{BW}(T) = \sigma_D(T)$. But $\sigma_D(T) = \sigma_{BF}(T) \cup$ acc $\sigma(T)$ by Theorem 2.5, hence λ is an isolated point of $\sigma(T)$. Since every algebraically M-hyponormal operator is isoloid by Lemma 4.5, $\lambda \in \pi_0(T)$.

Conversely, suppose that $\lambda \in \pi_0(T)$. Then λ is an isolated eigenvalue of T. Since λ is an isolated point of $\sigma(T)$, using the Riesz idempotent $E := \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1} d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum $T = T_1 \oplus T_2$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Since T is algebraically M-hyponormal, $p(T)$ is M-hyponormal for some nonconstant polynomial p. Since $\sigma(T_1) = \{\lambda_1\}$, we have $\sigma(p(T_1)) = p(\sigma(T_1)) = \{p(\lambda)\}$. Therefore $p(T_1) - p(\lambda)$ is quasinilpotent. Since $p(T_1)$ is M-hyponormal, it follows from Lemma 4.3 that $p(T_1) - p(\lambda) = 0$. Define $q(z) := p(z) - p(\lambda)$. Then $q(T_1) = 0$, and hence T_1 is algebraically M-hyponormal. Since $T_1 - \lambda$ is quasinilpotent and algebraically M-hyponormal, it follows from Lemma 4.4 that $T_1 - \lambda$ is nilpotent. Since $T - \lambda = (T_1 - \lambda) \oplus (T_2 - \lambda)$ is the direct sum of an invertible operator and a nilpotent operator, $T - \lambda$ is B-Weyl. Hence $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Therefore $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$, and hence $T \in g\mathcal{W}$.

Now let $f \in H(\sigma(T))$. Since T is algebraically M-hyponormal, it has SVEP. Therefore $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$. Since every algebraically M-hyponormal operator is isoloid by Lemma 4.5, it follows from Lemma 3.3 that $\sigma(f(T)) \setminus \pi_0(f(T)) = f(\sigma(T) \setminus \pi_0(T))$. Hence,

$$\begin{aligned}
\sigma(f(T)) \setminus \pi_0(f(T)) &= f(\sigma(T) \setminus \pi_0(T)) \\
&= f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)),
\end{aligned}$$

which implies that $f(T) \in g\mathcal{W}$. $\square$

Definition 4.8. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be paranormal if*

$$||Tx||^2 \leq ||T^2 x|| \quad \textit{for all } x \in \mathcal{H}, \; ||x|| = 1.$$

We say that $T \in \mathcal{B}(\mathcal{H})$ is algebraically paranormal if there exists a nonconstant complex polynomial p such that $p(T)$ is paranormal.

The following implications hold:

$$\begin{aligned}\text{hyponormal} &\Longrightarrow p\text{-hyponormal}\\ &\Longrightarrow \text{paranormal} \Longrightarrow \text{algebraically paranormal.}\end{aligned}$$

The following facts follow from Definition 4.8 and some well known facts about paranormal operators.

Lemma 4.9. *(i) If $T \in \mathcal{B}(\mathcal{H})$ is algebraically paranormal then so is $T - \lambda$ for every $\lambda \in \mathbb{C}$.*
(ii) If $T \in \mathcal{B}(\mathcal{H})$ is algebraically paranormal and $\mathcal{M} \subseteq \mathcal{H}$ is invariant under T, then $T|\mathcal{M}$ is algebraically paranormal.

In [14] we showed that if T is an algebraically paranormal operator then $f(T) \in \mathcal{W}$ for every $f \in H(\sigma(T))$. We can now extend this result to the generalized Weyl's theorem. To prove this we need several lemmas.

Lemma 4.10. *Let $T \in \mathcal{B}(\mathcal{H})$ be B-Fredholm. The following statements are equivalent:*
(i) T does not have SVEP at 0;
(ii) $a(T) = \infty$;
(iii) $0 \in \text{acc } \sigma_p(T)$.

Proof. Suppose that T is B-Fredholm. It follows from [5, Theorem 2.7] that T can be decomposed as

$$T = T_1 \oplus T_2 \ \ (T_1 \text{ Fredholm, } T_2 \text{ nilpotent}).$$

(i)$\Longleftrightarrow$(ii): Suppose that T does not have SVEP at 0. Since T_2 is nilpotent, T_2 has SVEP. Therefore T_1 does not have SVEP. Since T_1 is Fredholm, it follows from [1, Theorem 2.6] that $a(T) = \infty$.

Conversely, suppose that $a(T) = \infty$. Since T_2 is nilpotent, T_2 has finite ascent. Therefore $a(T_1) = \infty$. But T_1 is Fredholm, hence T_1 does not have SVEP by [1, Theorem 2.6].

(i)$\Longleftrightarrow$(iii): Suppose that T does not have SVEP at 0. Then T_1 does not have SVEP. Since T_1 is Fredholm, it follows from [1, Theorem 2.6] that $0 \in \text{acc } \sigma_p(T_1)$. Therefore $0 \in \text{acc } \sigma_p(T)$.

Conversely, suppose that $0 \in \text{acc } \sigma_p(T)$. Since T_2 is nilpotent, $0 \in \text{acc } \sigma_p(T_1)$. But T_1 is Fredholm, hence T_1 does not have SVEP by [1, Theorem 2.6]. Therefore T does not have SVEP. $\square$

Corollary 4.11. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is B-Fredholm with $i(T) > 0$. Then T does not have SVEP at 0.*

Proof. Suppose that T is B-Fredholm with $i(T) > 0$. Then by [5, Theorem 2.7], T can be decomposed by

$$T = T_1 \oplus T_2 \ \ (T_1 \text{ Fredholm, } T_2 \text{ nilpotent}).$$

Moreover, $i(T) = i(T_1)$. But $i(T) > 0$, hence $i(T_1) > 0$. Since T_1 is Fredholm, it follows from [15, Corollary 11] that T_1 does not have SVEP at 0. Therefore T does not have SVEP at 0. $\square$

Theorem 4.12. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is B-Fredholm. Then*

$$T^* \text{ does not have SVEP at } 0 \iff d(T) = \infty.$$

Moreover, if T and T^* have SVEP at 0 then T is B-Fredholm with index 0.

Proof. Since T is B-Fredholm, T can be decomposed by

$$T = T_1 \oplus T_2 \ \ (T_1 \text{ Fredholm, } T_2 \text{ nilpotent}).$$

But T_1 is Fredholm if and only if T_1^* is Fredholm, hence T is B-Fredholm if and only if T^* is B-Fredholm. Since T_1 is Fredholm, $a(T_1) = d(T_1^*)$. Also, since T_2 is nilpotent, $a(T_2) = d(T_2) =$

$a(T_2^*) = d(T_2^*)$. It follows from [23, Theorem 6.1] that

$$\begin{aligned} a(T^*) &= a(T_1^* \oplus T_2^*) \\ &= \max\{a(T_1^*), a(T_2^*)\} \\ &= \max\{d(T_1), d(T_2)\} \\ &= d(T_1 \oplus T_2) \\ &= d(T). \end{aligned}$$

Therefore by Lemma 4.10,

$$T^* \text{ does not have SVEP at } 0 \iff a(T^*) = \infty \iff d(T) = \infty.$$

Moreover, suppose that T and T^* have SVEP at 0. Then by Lemma 4.10, $a(T) = d(T) < \infty$, and hence T is B-Fredholm with index 0. □

Lemma 4.13. *(*[14, Lemmas 2.1, 2.2, 2.3]*) Let $T \in \mathcal{B}(\mathcal{H})$ be an algebraically paranormal operator. Then*
(i) If $\sigma(T) = \{\lambda\}$, then $T = \lambda$;
(ii) If T is quasinilpotent, then it is nilpotent;
(iii) T is isoloid.

Theorem 4.14. *Let $T \in \mathcal{B}(\mathcal{H})$ be an algebraically paranormal operator. Then $f(T) \in g\mathcal{W}$ for every $f \in H(\sigma(T))$.*

Proof. We first show that $T \in g\mathcal{W}$. Suppose that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda$ is B-Weyl but not invertible. Since T is an algebraically paranormal operator, there exists a nonconstant polynomial p such that $p(T)$ is paranormal. Since every paranormal operator has SVEP, $p(T)$ has SVEP. Therefore T has SVEP. It follows from Theorem 2.9 that $T \in g\mathcal{B}$. Therefore $\sigma_{BW}(T) = \sigma_D(T)$. But $\sigma_D(T) = \sigma_{BF}(T) \cup \text{acc } \sigma(T)$ by Theorem 2.5, hence λ is an isolated point of $\sigma(T)$. Since every algebraically paranormal operator is isoloid by Lemma 4.13, $\lambda \in \pi_0(T)$.

Conversely, suppose that $\lambda \in \pi_0(T)$. Let $P := \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1} d\mu$ be the associated Riesz idempotent, where D is an open disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum $T = T_1 \oplus T_2$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Now we consider two cases:

Case I. Suppose that $\lambda = 0$. Then T_1 is algebraically paranormal and quasinilpotent. It follows from Lemma 4.13 that T_1 is nilpotent. Therefore T is the direct sum of an invertible operator and nilpotent, and hence T is B-Weyl by [6, Lemma 4.1]. Thus, $0 \in \sigma(T) \setminus \sigma_{BW}(T)$.

Case II. Suppose that $\lambda \neq 0$. Since T is algebraically paranormal, $p(T)$ is paranormal for some nonconstant polynomial p. Since $\sigma(T_1) = \{\lambda_1\}$, we have $\sigma(p(T_1)) = p(\sigma(T_1)) = \{p(\lambda)\}$. Therefore $p(T_1) - p(\lambda)$ is quasinilpotent. Since $p(T_1)$ is paranormal, it follows from Lemma 4.13 that $p(T_1) - p(\lambda) = 0$. Define $q(z) := p(z) - p(\lambda)$. Then $q(T_1) = 0$, and hence T_1 is algebraically paranormal. Since $T_1 - \lambda$ is quasinilpotent and algebraically paranormal, it follows from Lemma 4.13 that $T_1 - \lambda$ is nilpotent. Since $T - \lambda = \begin{pmatrix} T_1 - \lambda & 0 \\ 0 & T_2 - \lambda \end{pmatrix}$ is the direct sum of an invertible operator and nilpotent, $T - \lambda$ is B-Weyl. Therefore $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Thus $T \in g\mathcal{W}$.

Let $f \in H(\sigma(T))$. Since T is algebraically paranormal, it has SVEP. Therefore $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$. Also, since T is algebraically paranormal, it follows from Lemma 4.13 that T is isoloid. Therefore by Lemma 3.3,

$$\sigma(f(T)) \setminus \pi_0(f(T)) = f(\sigma(T) \setminus \pi_0(T)).$$

Hence

$$\sigma(f(T)) \setminus \pi_0(f(T)) = f(\sigma(T) \setminus \pi_0(T)) = f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)),$$

which implies that $f(T) \in g\mathcal{W}$. $\square$

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Department of Mathematics, University of Iowa, Iowa City, IA 52242-1419.
E-mail address: `rcurto@math.uiowa.edu`

Department of Mathematics, Kyunghee University, Seoul, Korea 130-701.
E-mail address: `ymhan2004@khu.ac.kr`