EULERIAN CHAINS

Definition *A chain which goes through all edges of a graph G, and uses each edge of G at most once is called an* ***Eulerian Chain****. An* ***Eulerian Closed Chain*** *is an Eulerian Chain which is closed.*

Remark *An Eulerian Closed Chain need* ***NOT*** *be a circuit, i.e simple closed. That is, an Eulerian chain may pass through some vertices (in fact all with degree ≥ 4) more than once. If "Eulerian Circuit" or "Eulerian Cycle" is used inadvertently in the notes, then they must be replaced by "Eulerian Closed Chain" or "Eulerian Closed Path", respectively.*

Theorem *(EULER) Let G be a connected multi-graph, with no loops.*

(a) There exists an ***Eulerian Closed Chain*** *in G $\Leftrightarrow$ Degrees of all vertices of G are even.*

(b) There exists an Eulerian Chain in G $\Leftrightarrow$ The number of odd degree vertices in G is either 0 or 2.

EULERIAN CLOSED CHAIN ALGORITHM

GIVEN: A connected graph G with all even degree vertices

TO FIND: An Eulerian Closed Chain, that is a closed chain which goes through all edges of the graph, and uses each edge of the graph at most once.

1. Choose any vertex, and label it u_0. Choose any edge starting at u_0, label it E_1 and go to the other end point v_1 of E_1.

If there is no such edge E_1, then G is a single point, and process is finished.

2. Assume that first k edges $E_1, E_2, \ldots, E_k$ of an Eulerian chain has been constructed, which is starting at u_0, passing through $v_1, v_2, \ldots v_{k-1}$ and ending at v_k.

Go to end point v_k of E_k, and consider all of the remaining unlabeled edges starting at v_k :

If there is an unlabeled edge, choose one, label it E_{k+1} and its other end point v_{k+1}, and repeat step 2, until it can't be done anymore.

If there is not any unlabeled edges left starting at v_k, then v_k must be the initial vertex u_0. Label the closed chain $E_1, E_2, \ldots E_k$ with C_1 and go to step 3.

The reason that this process must go on through any vertex but the initial vertex, is that all degrees are even, and any time one passes through a vertex, the number of unlabeled edges touching that vertex goes down by 2. Hence, if it is possible to reach a vertex other than the initial vertex, then there is a way out of that vertex.

3. If C_1 used all of the edges of G, then the process is finished.

If C_1 did not use all of the edges of G, then let G_2 be the graph obtained from G by removing all the edges (not the vertices) used by C_1 and then removing all isolated vertices.

At this point, we know that $C_1 \cap G_2 \neq \emptyset$, since otherwise $C_1 \cap G_2 = \emptyset$ implies that G has at least 2 components, C_1 and G_2. However, recall that G is connected. Caution: G_2 may not be connected.

4. Choose any vertex u_1 of $C_1 \cap G_2$.

On G_2 starting from the initial vertex u_1 repeat steps 1, 2 (several times) until you reach step 3.

Label the new closed chain (from u_1 to itself) you obtained with $\tilde{C}_2$.

Construct a closed chain C_2 as follows: Start at u_0. Follow C_1 to u_1, follow all of $\tilde{C}_2$ starting and ending at u_1, follow the rest of C_1 to u_0.

5. If C_2 used all of the edges of G, then the process is finished.

If C_2 did not use all of the edges of G, then let G_3 be the graph obtained from G by removing all the edges (not the vertices) used by C_2 and then removing all isolated vertices. Then repeat step 4, by choosing a vertex u_2 of $C_2 \cap G_3$ and so on. Repeat step 5, until you obtain an Eulerian chain C_m which uses all edges of G.

This process must finish after a finite repetition of step 5, since $0 \leq e(G_3) < e(G_2) < e(G) < \infty$, where $e(G)$ is the number of edges of G.

HW

EULERIAN CHAINS, etc

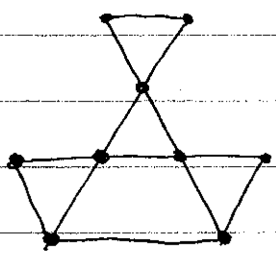
For each of the following graphs (or digraphs)

(a) Find an Eulerian closed chain if there is one,

(b) " " " chain if " " ", or

(c) If neither is possible, state it so & explain why.

(1)



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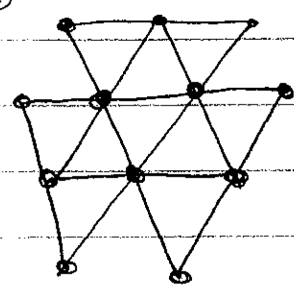

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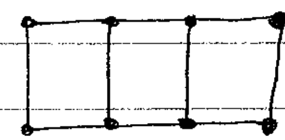

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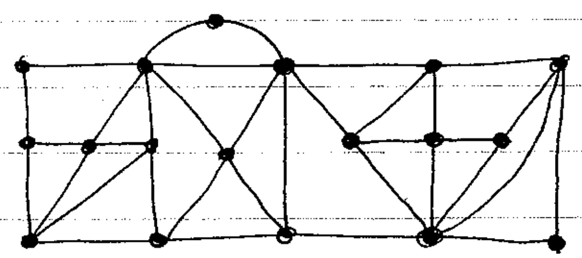

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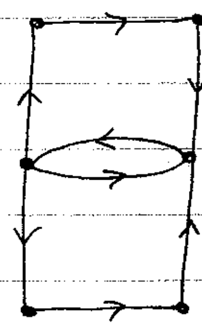

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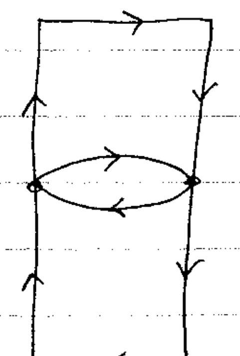

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