

Characterization of PF rings by the Finite Topology on duals of R Modules

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March 10, 2003

Abstract : In this paper we study the properties of the finite topology on the dual of a module over an arbitrary ring. We aim to give conditions when certain properties of the field case can still be found here. Investigating the correspondence between the closed submodules of the dual M^* of a module M and the submodules of M , we prove some characterizations of PF rings: the up stated correspondence is an anti isomorphism of lattices if and only if R is a PF ring.

Key words and phrases : PF rings, Finite Topology

Mathematics Subject Classification (2000) : 16D50, 16L99

1 Introduction

Let R be an arbitrary (non commutative) ring. We will use the notations $\text{Hom}_R(M, N)$ for the set of R module morphisms from M to N for right modules M, N and ${}_R\text{Hom}(M, N)$ respectively for left modules M, N . Also we use $M^* = \text{Hom}_R(M, R)$ for any right module M and ${}^*M = {}_R\text{Hom}(M, R)$ for a left module M .

Given two right R modules M and N , recall that the finite topology on $\text{Hom}_R(M, N)$ is the linear topology for which a basis of open neighborhoods for 0 is given by the sets $\{f \in \text{Hom}_R(M, N) \mid f(x_i) = 0, \forall i \in \{1, \dots, n\}\}$, for all finite sets $\{x_1, \dots, x_n\} \subseteq M$. This is actually the topology induced on $\text{Hom}_R(M, N)$ from $\text{Hom}_{\text{Set}}(M, N) = N^M$ which is a product of topological spaces, where N is the topological discrete space on the set N . For an arbitrary set $X \subseteq M$ we denote by $X^\perp = \{f \in \text{Hom}_R(M, N) \mid f|_X = 0\}$. If $W \leq \text{Hom}_R(M, N)$ is a subgroup with M and N left R modules we denote $W^\perp = \{x \in N \mid f(x) = 0, \forall f \in W\}$.

The Finite Topology on the dual of a vector space (or the dual of a module) is an important tool in different topics of algebra, such as coalgebras and Hopf algebras, the theory of graded rings and modules, or more general, the theory of corings. For example, the semiperfectness property for

coalgebras (or even corings) is strongly connected to properties of the finite topology of the dual algebra (or the dual ring); the topological tool allows the connection of a series of properties of coalgebras to semiperfect coalgebras. Furthermore, the topological aspects of Hopf algebras allow characterizations of (co)Frobenius Hopf algebras as Hopf algebras with nonzero integrals.

Regarding the phenomenon for the field case, we find a series of properties connecting the subspaces of a vector space V and the subspaces of V^* , via the taking of orthogonal (X^\perp), properties that have a dual flavor. Some of these properties have also been studied in literature before (see for example [2] and [3]), where a different notation is used for X^\perp . We still state some of these results for completion of the presentation, some of them in a slightly more general setting or some of them with short or no proof (as for example proposition 1, 1, 3, 4). But we are mainly interested in the topological aspects, namely the set of closed submodules of the dual M^* of a module M .

A right PF ring is a ring that is right cogenerator and right self-injective, that is, it is an injective cogenerator of the category \mathcal{M}_R . This is known ([1],[9],[10] cited by C.Faith [7]) to be equivalent to the fact that the ring R is right self-injective and has essential right socle, and also, to the fact that R decomposes as a finite direct sum $\bigoplus_{i=1}^n e_i R$ with e_i idempotents such that $e_i R$ are indecomposable injective with simple socle, $i = 1, \dots, n$. It is also known [8] that a ring R is both left and right PF (and called simply PF) if and only if R is right PF and left self-injective, and this is also equivalent to R being right and left cogenerator. A left PF ring need not necessarily be right PF, as an example in [5] shows.

A deeper investigation of the properties of the finite topology shows that the dual character of the connection between closed subspaces (or submodules) of V^* and subspaces (or submodules) of V is given by some properties of the base ring R , namely R self-injective and R cogenerator, leading us to the study of PF rings. Equivalent characterisations of PF rings are found then, namely PF rings are exactly those rings for which the lattice of the submodules of any right (or, equivalently, any left) module M is anti-isomorphic to the lattice of the closed submodules of M^* ; equivalently, for finitely generated M 's, the lattice the of submodules of M is anti-isomorphic to the lattice the of submodules of M^* . This is the main result of the paper, and is given by Theorem 1. As a corollary, some known characterizations of PF rings given by Kato in [8] are obtained, as well as some interesting facts on PF rings (Corollary 2, 3, 4).

2 Preliminary results

Denoting by $\langle X \rangle_R$ the R submodule generated by X , we obviously have $(\langle X \rangle_R)^\perp = X^\perp$, so we will work with finitely generated submodules $F \leq M$ and the basis of open neighborhoods $\{F^\perp \mid F \leq M \text{ finitely generated}\}$. Also for left R modules X and Y and $U \leq X$ a submodule of X we will denote $U_{R\text{Hom}(M,N)}^\perp$ or simply $U^\perp = \{g \in {}_R\text{Hom}(X, Y) \mid g|_U = 0\}$ when there is no danger of confusion. If N is an R bimodule then we consider the left R module structure on $\text{Hom}_R(M, N)$ given by $(r \cdot f)(x) = rf(x)$, for all $x \in M, f \in \text{Hom}_R(M, N), r \in R$. If W is a (left) submodule in $\text{Hom}_R(M, N)$, then W^\perp is a (right) submodule of M . For any right module M we denote by Φ_M the right R modules morphism

$$M \xrightarrow{\Phi_M} {}^*(M^*)$$

defined by $\Phi_M(m)(f) = f(m)$, for all $f \in M^*$ and all $m \in M$. Then Φ is a functorial morphism from $id_{\mathcal{M}_R}$ to the functor ${}^*((-)^*)$.

We first extend some properties of the finite topology from the field case to more general cases and give some properties of the finite topology on duals of modules over PF rings. In Section 2 we obtain equivalent characterizations of PF rings.

Proposition 1. *Let M, N be R modules.*

- (i) *If $X \subseteq Y$ are submodules of M then $Y^\perp \leq X^\perp$.*
- (ii) *If $U \subseteq V$ are subgroups of $\text{Hom}_R(M, N)$ then $V^\perp \leq U^\perp$.*

Lemma 1. *For M, N right R modules we have:*

- (i) *If $X \leq M$ is a submodule of M then $(X^\perp)^\perp \supseteq X$ and if we denote $\bar{0}$ the class of 0 in M/X then we have $(\{\bar{0}\}^\perp)^\perp = (X^\perp)^\perp/X$. If N is an injective cogenerator of \mathcal{M}_R then the equality $(X^\perp)^\perp = X$ holds.*
- (ii) *If $Y \leq \text{Hom}_R(M, N)$ is a (left) submodule of $\text{Hom}_R(M, N)$ then $(Y^\perp)^\perp \supseteq \bar{Y}$ (\bar{Y} is the closure of Y in $\text{Hom}_R(M, N)$). If $N = R$ and R is a left PF ring (${}_R R$ is injective and a cogenerator of ${}_R \mathcal{M}$) then the equality $(Y^\perp)^\perp = \bar{Y}$ holds for all modules M and (left) submodules $Y \leq M^*$.*

Proof. (i) If $x \in X$ then take $f \in X^\perp$; then $f(x) = 0$ as $f|_X = 0$. We get that $f(x) = 0, \forall f \in X^\perp$ so $x \in (X^\perp)^\perp$. Moreover, $\bar{x} \in (\{\bar{0}\}^\perp)^\perp$ if and only if $\tilde{h}(\bar{x}) = 0, \forall \tilde{h} : M/X \rightarrow N$, equivalent to $h(x) = 0, \forall h \in X^\perp$, i.e. $x \in (X^\perp)^\perp$.

Suppose now N is an injective cogenerator of \mathcal{M}_R and take $x \in (X^\perp)^\perp$. If $x \notin X$ then there is $f : M/X \rightarrow N$ such that $f(\hat{x}) \neq 0$ (\hat{x} is the image

of x in M/X via the canonic morphism $\pi : M \rightarrow M/X$. Then there is $g = f \circ \pi$, $g \in \text{Hom}_R(M, N)$ such that $g|_X = 0$ ($g \in X^\perp$) and $g(x) \neq 0$, showing that $x \notin (X^\perp)^\perp$, a contradiction.

(ii) Let $f \in \bar{Y}$ and take $x \in Y^\perp$. Then there is $g \in Y$ such that $f(x) = g(x)$. But $g(x) = 0$ because $x \in Y^\perp$ so $f(x) = 0$. Thus $f|_{Y^\perp} = 0$ and $f \in (Y^\perp)^\perp$. For the converse, first we see that ${}_R R$ injective implies that for all finitely generated right R modules F we have that $F \xrightarrow{\Phi_F} {}^*(F^*)$ is an epimorphism. Take $\pi : P = R^n \rightarrow F$ an epimorphism in \mathcal{M}_R . Then we have a monomorphism $0 \rightarrow P^* \rightarrow F^*$ in ${}_R \mathcal{M}$, and as ${}_R R$ is injective we obtain an epimorphism of right modules ${}^*(P^*) \xrightarrow{{}^*(\pi^*)} {}^*(F^*) \rightarrow 0$. Because Φ is a functorial morphism then we have the commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{\pi} & F & \longrightarrow & 0 \\ \Phi_P \downarrow & & \downarrow \Phi_F & & \\ {}^*(P^*) & \xrightarrow{{}^*(\pi^*)} & {}^*(F^*) & \longrightarrow & 0 \end{array}$$

showing that Φ_F is surjective, as $\Phi_P = \Phi_{R^n}$ is an isomorphism. Now to prove the desired equality, take $f \in (Y^\perp)^\perp$, $(f_i)_{i \in I}$ a family of generators of the left R module Y , and $F < M$ a finitely generated submodule of M . Then $f_i|_M \in F^*$ and if $f|_F \notin \langle f_i|_F \mid i \in I \rangle$ then as ${}_R R$ is an injective cogenerator of ${}_R \mathcal{M}$ we can find a morphism of left R modules $\phi : F^* \rightarrow R$ such that $\phi(f_i) = 0$, $\forall i \in I$ and $\phi(f) \neq 0$. But as Φ_F is surjective, we can then find $x \in F$ such that $\phi = \Phi(x)$ and then $f_i(x)\Phi(x)(f_i) = \phi(f_i) = 0$, $\forall i \in I$, showing that $x \in Y^\perp$ and $f(x) = \Phi(x)(f) = \phi(f) \neq 0$ which contradicts the fact that f belongs to $(Y^\perp)^\perp$. Thus we must have $f|_F \in \langle f_i|_F \mid i \in I \rangle$ so there is $(r_i)_{i \in I}$ a family of finite support such that $f|_F = \sum_{i \in I} r_i(f_i|_F) = (\sum_{i \in I} r_i f_i)|_F$. This last relation shows that $f \in \bar{Y}$. \square

Corollary 1. *If R is a PF ring (left and right) then for any right (or left) R module M and $Y < M^*$ we have that Y is dense in M^* if and only if $Y^\perp = 0$.*

Proposition 2. *Let M be a right R module.*

- (i) *If $X \leq M$ then we have $((X^\perp)^\perp)^\perp = X^\perp$ and X^\perp is closed.*
- (ii) *If $Y \leq \text{Hom}_R(M, N)$ then $((Y^\perp)^\perp)^\perp = Y^\perp$.*

Proof. " \subseteq " from (i) and (ii) follow from Proposition 1 and Lemma 1.

(i) " \supseteq " Let $f \in X^\perp$. Take $x \in (\overline{X^\perp})^\perp$; then $f(x) = 0$ so $f \in ((X^\perp)^\perp)^\perp$. To show that X^\perp is closed take $f \in \overline{X^\perp}$ and $x \in X$. Then there is $g \in X^\perp$ such that $g(x) = f(x)$ so $f(x) = 0$ ($x \in X$). We obtain that $f|_X = 0$ so $f \in X^\perp$.

(ii) " \supseteq " Let $x \in Y^\perp$. If $f \in (Y^\perp)^\perp$ then $f|_{Y^\perp} = 0$ so $f(x) = 0$ showing that $x \in ((Y^\perp)^\perp)^\perp$. \square

Proposition 3. Let M, N be right R modules and $(X_i)_{i \in I}$ a family of submodules of M . Then

$$(i) \left(\sum_{i \in I} X_i\right)^\perp = \bigcap_{i \in I} X_i^\perp.$$

(ii) $\left(\bigcap_{i \in I} X_i\right)^\perp \supseteq \sum_{i \in I} X_i^\perp$. If I is finite and N is injective then equality holds.

Proof. (i) $f \in \left(\sum_{i \in I} X_i\right)^\perp \Leftrightarrow f|_{\sum_{i \in I} X_i} = 0 \Leftrightarrow f|_{X_i} = 0, \forall i \in I \Leftrightarrow f \in X_i^\perp, \forall i \in I \Leftrightarrow f \in \bigcap_{i \in I} X_i^\perp$.

(ii) " \supseteq " is obvious, for Proposition 1 shows that $X_i^\perp \subseteq \bigcap_{j \in I} X_j^\perp, \forall i \in I$.

For the converse it is enough to prove the equality for two submodules X, Y of M . Denote $\pi : M \rightarrow M/X \cap Y, p : M \rightarrow M/X, q : M \rightarrow M/Y$ the canonical morphisms. If $f \in \text{Hom}_R(M, N)$ such that $f|_{X \cap Y} = 0$ then denote $\bar{f} : M/X \cap Y \rightarrow N$ the factorization of f ($f = \bar{f} \circ \pi$) and $i : M/X \cap Y \rightarrow M/X \oplus M/Y$ the injection $i(\pi(x)) = (p(x), q(x)), \forall x \in M$. Then the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \frac{M}{X \cap Y} \xrightarrow{i} \frac{M}{X} \oplus \frac{M}{Y} \\ & & \downarrow \bar{f} \quad \swarrow h = \bar{u} \oplus \bar{v} \\ & & N \end{array}$$

is completed commutatively by h . Then $h = \bar{u} \oplus \bar{v}$, with $\bar{u} \in \text{Hom}_R(M/X, N)$ and $\bar{v} \in \text{Hom}_R(M/Y, N)$, such that $h(p(x), q(x)) = \bar{u}(p(x)) + \bar{v}(q(x))$. Taking $u = \bar{u} \circ p$ and $v = \bar{v} \circ q$ we have $u \in X^\perp, v \in Y^\perp$ and $f(x) = \bar{f}(\pi(x)) = h(i(\pi(x))) = h(p(x), q(x)) = \bar{u}(p(x)) + \bar{v}(q(x)) = u(x) + v(x), \forall x \in M$, so $f \in X^\perp + Y^\perp$. \square

Proposition 4. Let M, N be right R modules and $(Y_i)_{i \in I}$ be a family of submodules of $\text{Hom}_R(M, N)$. Then:

$$(i) \left(\sum_{i \in I} Y_i \right)^\perp = \bigcap_{i \in I} Y_i^\perp.$$

(ii) $\left(\bigcap_{i \in I} Y_i \right)^\perp \supseteq \sum_{i \in I} Y_i^\perp$. If $N = R$ and R is a PF ring (both left and right PF) and Y_i are closed subsets of $M^* = \text{Hom}_R(M, R)$ then the equality holds: $\left(\bigcap_{i \in I} Y_i \right)^\perp = \sum_{i \in I} Y_i^\perp$.

Proof. (i) Obvious.

(ii) "⊇" similar to (ii)"⊇" of the previous proposition. For the converse inclusion, take $(Y_i)_{i \in I}$ a family of submodules of M^* . Then

$$\begin{aligned} \sum_{i \in I} Y_i^\perp &= \left(\left(\sum_{i \in I} Y_i^\perp \right)^\perp \right)^\perp \quad (\text{from Lemma 1 : } R \text{ is right PF}) \\ &= \left(\bigcap_{i \in I} (Y_i^\perp)^\perp \right)^\perp \quad (\text{from Proposition 3}) \\ &= \left(\bigcap_{i \in I} Y_i \right)^\perp \quad (\text{Lemma 1 : } Y_i \text{ are closed and } R \text{ is left PF}) \end{aligned}$$

□

Remarks (i) The equality in Proposition 3 does not hold for infinite sets. Let V be an infinite dimensional space with a countable basis indexed by the set of natural numbers: $(e_n)_{n \in \mathbf{N}}$. Put $V_n = \langle e_k \mid k \geq n \rangle$. Then we can easily see that $\bigcap_{n \in \mathbf{N}} V_n = 0$ so $\left(\bigcap_{n \in \mathbf{N}} V_n = 0 \right)^\perp = V^*$. Let $f \in V^*$ be the function equal to 1 on all the e_n -s. Then as $V_n^\perp < V_m^\perp, \forall n < m$, we have that $f \in \sum_{n \in \mathbf{N}} V_n^\perp \Leftrightarrow \exists n \in \mathbf{N}$ such that $f \in V_n^\perp$ which is impossible as $f(e_n) = 0, \forall n$. We obtain $\bigcap_{n \in \mathbf{N}} V_n \supset \sum_{n \in \mathbf{N}} V_n^\perp$ a strict inclusion.

(ii) The equality in Proposition 4 does not hold for non-closed sets. Let again V be a vector space with a countable basis $B = (e_n)_{n \in \mathbf{N}}$. Denote by e_n^* the linear map equal to 1 on e_n and 0 on the other elements of the basis B and by f^* the linear map equal to 1 on all the e_n -s. Take $H = \langle e_n^* \mid n \in \mathbf{N} \rangle$ and $L = \langle f^*, e_n^* \mid n \in \mathbf{N}^* \rangle$. Then we can easily see that $H^\perp = 0, L^\perp = 0$ and $H \cap L = \langle e_n^* \mid n \in \mathbf{N}^* \rangle$, so $H^\perp + L^\perp = 0$, but $(H \cap L)^\perp = \langle f^*, e_n^* \mid n \in \mathbf{N} \rangle^\perp = \langle e_0 \rangle$, thus $H^\perp + L^\perp \neq (H \cap L)^\perp$.

(iii) Given the same vector space, we give an example of a family of dense subspaces of V^* whose intersection is 0. For $p \in \mathbf{N}$ let $H_p = \langle e_n^* + e_{n+1}^* + \dots + e_{n+p}^* \mid n \in \mathbf{N} \rangle$. Then a short computation shows that $H_n^\perp = 0$ showing that H_n is closed in V^* . But $\bigcap_{n \in \mathbf{N}} H_n = 0$, because if $f = \sum_{i=1}^m \lambda_i e_i^* \in \bigcap_{n \in \mathbf{N}} H_n \subset H_0$, then $f \in H_{m+1}$ which shows that if $f \neq 0$, than it can be written as a

linear combination of e_i^* in which at least one of the e_i^* has $i > m$. This is impossible as the e_n^* -s are independent.

3 The Finite Topology vs PF Rings

If R is a ring then we have $(R^n)^* = \text{Hom}_R(R, R) \simeq {}_R R^n$. So we can identify R submodules of the right dual of R^n with left submodules of ${}_R R$ and vice versa. For all $x = (x_1, \dots, x_n) \in R^n$ we denote by $\varphi_x : R^n \rightarrow R$ the morphism of right R modules $\varphi_x(r_1, \dots, r_n) = \sum_{i=1}^n x_i r_i$ and by ψ_x the morphism

of left modules defined by $\psi_x(r_1, \dots, r_n) = \sum_{i=1}^n r_i x_i, \forall (r_1, \dots, r_n) \in R^n$.

Also because of the isomorphism $(R^n)^* \simeq {}_R R^n, x \mapsto \varphi_x$, we will denote by $I^\perp = \{x \in R^n \mid \varphi_x(r) = 0, \forall r \in I\}$ if I is a right submodule of R^n and similarly for left submodules X of $R^n, X^\perp = \{x \in R^n \mid \psi_x(r) = 0, \forall r \in X\}$.

Over a vector space V there is an anti isomorphism of lattices between the lattice of closed subspaces of V^* and the subspaces of V given by $X \mapsto X^\perp, \forall X \leq V$. We have the obvious

Proposition 5. *For a right module M the following are equivalent:*

(i) *The applications $M \geq X \mapsto X^\perp \leq M^*$ and $M^* \geq Y \mapsto Y^\perp \leq M$ between the lattice of the submodules of M and the lattice of the closed submodules of M^* are inverse anti-isomorphisms of lattices.*

(ii) *$(X^\perp)^\perp = X, \forall X \leq M$ and $(Y^\perp)^\perp = \overline{Y}, \forall Y \leq M^*$.*

(iii) *$(X^\perp)^\perp = X, \forall X \leq M$ and $(Y^\perp)^\perp = Y, \forall Y \leq M^*, Y$ closed.*

(iv) *The applications of (i) are inverse to each other.*

If F is a finitely generated right R module then every submodule of F^* is closed, as if Y is a left submodule of F^* and $f \in \overline{Y}$, taking $\{x_1, \dots, x_n\}$ the a system of generators of F , there is $g \in Y$ such that $g(x_i) = f(x_i)$, for all i , so $f = g \in Y$. Also it is easy to see that R^n has orthogonal equivalence as right module if and only if it has orthogonal equivalence as left module, and this is equivalent to $(I^\perp)^\perp = I, \forall I \leq R^n$ and $(X^\perp)^\perp = X, \forall X \leq R^n$.

Definition 1. *We will say that a right R module M has orthogonal equivalence (or orthogonal isomorphism, or shortly M has \perp equivalence) if the equivalent statements of Proposition 5 hold. The ring R will be called with \perp equivalence if R_R (or equivalently ${}_R R$) is a module with orthogonal equivalence.*

Proposition 6. *Let M be a right R module and X a submodule of M . Then we have the exact sequence*

$$0 \longrightarrow (0^\perp)^\perp \longrightarrow M \xrightarrow{\Phi_M} {}^*(M^*)$$

Proof. For $x \in M$ we have $\Phi_M(x) = 0 \Leftrightarrow f(x) = 0, \forall f \in M^*$ and this is equivalent to $x \in (M^*)^\perp = (0^\perp)^\perp$, thus $\ker \Phi_M = (0^\perp)^\perp$. \square

Proposition 7. (i) *For an R module M we have $(0^\perp)^\perp = 0$ if and only if M is R cogenerated, i.e. there is a monomorphism $M \hookrightarrow R^I$ for some set I . Following [7], in this case M is called torsionless module.*

(ii) *If \mathcal{C} is a class of right R modules which is closed under quotients then the following are equivalent:*

(a) $(X^\perp)^\perp = X$ for all M in \mathcal{C} , $X < M$.

(b) $(0^\perp)^\perp = 0$ for all M in \mathcal{C} .

(c) Any $M \in \mathcal{C}$ is cogenerated by R .

(d) Φ_M is a monomorphism for every M in \mathcal{C} .

Proof. (i) If $(0^\perp)^\perp = 0$ then take $I = M^*$ and $M \xrightarrow{i} R^I$, $i(x) = (f(x))_{f \in I}$; then of course i is a monomorphism as $i(x) = 0$ if and only if $f(x) = 0, \forall f \in I = M^*$ i.e. $x \in (0^\perp)^\perp = 0$. Conversely, given a monomorphism $M \xrightarrow{i} R^I$, taking π_j the canonical projections for all $j \in I$, we obtain the morphisms $f_j = \pi_j \circ i \in M^*$ and then $x \in (0^\perp)^\perp = (M^*)^\perp$ implies $f_j(x) = 0, \forall j \in I$, i.e. $i(x) = 0$ so $x = 0$, as i is injective. Thus $(0^\perp)^\perp = 0$.

(ii) (b) \Leftrightarrow (c) by (i). (a) \Leftrightarrow (b) follows as \mathcal{C} is closed under quotient objects and denoting $\bar{0}$ the zero element of $M/X \in \mathcal{C}$ we have $(\{\bar{0}\}^\perp)^\perp = (X^\perp)^\perp$ from Lemma 1. Equivalence with (d) follows from Proposition 6 \square

Proposition 8. *Suppose R_R is a module with \perp equivalence. Then R contains all left simple modules and all right simple modules (up to an isomorphism; this is called a right - and left- Kasch ring).*

Proof. It is easy to see that for every right ideal I of R we have the isomorphism of left R modules $(\frac{R}{I})^* \simeq I^\perp$, given by $I^\perp \ni f \mapsto f \circ \pi \in (\frac{R}{I})^*$, with $\pi : R \longrightarrow R/I$ the canonical projection. Then if S is simple right module there is a maximal right ideal $M < R$ and an isomorphism $S \simeq \frac{R}{M}$. Then $S^* \simeq (\frac{R}{M})^* \simeq M^\perp \neq 0$ because if $M^\perp = 0$ then $M = (M^\perp)^\perp = 0^\perp = R$, which contradicts the maximality of M . In a similar way one can see that R contains all the isomorphism types of left R modules. \square

We shall say a right (or left) R module is n generated if it has a system of n generators.

Lemma 2. *Let X be a right R module such that every monomorphism $i : X \hookrightarrow M$ with the property that $M/\text{Im } i$ is 1-generated splits. Then X is an injective module.*

Proof. Let M be a right R module such that $X < M$ (we identify X with its image in M) and suppose $X \neq M$. Let $\mathcal{L} = \{Y < M \mid Y \neq 0 \text{ and } X \cap Y = 0\}$. Then $\mathcal{L} \neq \emptyset$, because if $x \in M \setminus X$ then as $(X + xR)/X \neq 0$ is 1-generated then the hypothesis shows that there is $Y < X + xR$ such that $X + Y = X + xR$ and then $Y \neq 0$ as $x \notin X$, so $Y \in \mathcal{L}$. We can easily see that \mathcal{L} is inductive, because if $(Y_i)_{i \in I}$ is a totally ordered family of elements of \mathcal{L} then $\bigcup_{i \in I} Y_i$ is its majorant in \mathcal{L} . Take N a maximal element of \mathcal{L} and suppose $X + N \neq M$. Then there is $x \in M \setminus (X + N)$ and as $(X + N + xR)/(X + N)$ is 1-generated, by the hypothesis we can find $Y < M$ such that $X + N + Y = X + N + xR$ and $(X + N) \cap Y = 0$. An easy computation shows now that $(N + Y) \cap X = 0$ and so $N + Y = N$ by the maximality of N . Thus we obtain $X + N + Y = X + N = X + N + xR$ which is a contradiction, because $x \notin X + N$. We find that X is a direct summand in M for every module M such that $X \hookrightarrow M$, so X is injective in \mathcal{M}_R . \square

Proposition 9. *Let R be a ring with \perp equivalence. If $R \xrightarrow{j} X$ is a monomorphism of right (left) R modules and X is R cogenerated then j splits.*

Proof. Consider $X \xrightarrow{\sigma} R^I$ a monomorphism and let $(x_i)_{i \in I} = \sigma(j(1))$. Then we have $(x_i r)_{i \in I} = \sigma(j(1))r = \sigma(j(r))$ and as j, σ are injective we see that $x_i r = 0, \forall i \in I$ if and only if $r = 0$. This shows that $\bigcap_{i \in I} Rx_i^\perp = 0$.

Then we have $0 = \bigcap_{i \in I} Rx_i^\perp = (\sum_{i \in I} Rx_i)^\perp$ (by Proposition 3), so $\sum_{i \in I} Rx_i = ((\sum_{i \in I} Rx_i)^\perp)^\perp = 0^\perp = R$. Then we find that there is F a finite subset of I such that $\sum_{i \in F} Rx_i = R$, thus there are $(y_i)_{i \in F} \in R$ such that $\sum_{i \in F} y_i x_i = 1$.

Now if we denote by π_F the projection of R^I on R^F , $\pi_F((r_i)_{i \in I}) = (r_i)_{i \in F}$ and by $y = (y_i)_{i \in F} \in R^F = R^{(F)}$, then $\varphi_y(\pi_F(\sigma(j(r)))) = \varphi_y(\pi_F((x_i r)_{i \in I})) = \varphi_y((x_i r)_{i \in F}) = \sum_{i \in F} y_i x_i r = r$, so $\varphi_y \circ \pi_F \circ \sigma \circ j = id_R$, showing that the morphism of right modules $\varphi_y \circ \pi_F \circ \sigma : X \longrightarrow R$ is a split for j . \square

Lemma 3. *R^n has orthogonal equivalence (as left or right R module) if and only if every n generated right (or left) module has orthogonal equivalence.*

Proof. Suppose R^n has \perp equivalence. Let $F = R^n/X$ be a right n generated R modules and $\pi : R^n \rightarrow F$ the canonical projection. For each $g \in X^\perp$ ($X < R^n$) we denote by $\bar{g} \in F^*$ the (unique) morphism for which $\bar{g} \circ \pi = g$ and with $\hat{x} = \pi(x)$ - the class of an element $x \in R^n$. Now we see that if $Y < F^*$ and $Z = \{\alpha \circ \pi \mid \alpha \in Y\}$, then $Y = \{\bar{g} \mid g \in Z\}$, $Y^\perp = \{\hat{x} \mid \bar{g}(\hat{x}) = 0, \forall g \in Z\} = Z^\perp/X$ ($Z \subseteq X^\perp$ so $Z^\perp \supseteq (X^\perp)^\perp = X$) and $(Y^\perp)^\perp = \{\bar{g} \mid \bar{g}(\hat{x}) = 0, \forall \hat{x} \in Z^\perp/X\} = \{\bar{g} \mid g(x) = 0, \forall x \in Z^\perp\} = \{\bar{g} \mid g \in (Z^\perp)^\perp = Z\} = Y$.

Now if $Y < F$ and $Z = \pi^{-1}(Y)$ then $Y^\perp = \{\bar{g} \mid \bar{g}(\hat{x}) = 0, \forall \hat{x} \in Y\} = \{\bar{g} \mid g(x) = 0, \forall x \in Z\} = \{\bar{g} \mid g \in Z^\perp\}$ and $(Y^\perp)^\perp = \{\hat{x} \mid \bar{g}(\hat{x}) = g(x) = 0, \forall g \in Z^\perp\} = \{\hat{x} \mid x \in (Z^\perp)^\perp = Z\} = Y$. \square

Theorem 1. *The following assertions are equivalent:*

- (i) *Every right R module has \perp equivalence.*
- (ii) *Every finitely generated module has \perp equivalence.*
- (iii) *Every left R module has \perp equivalence.*
- (iv) *Every finitely generated module has \perp equivalence.*
- (v) *R is a PF ring (both left and right).*
- (vi) *$(X^\perp)^\perp = X$ for all $X < M$ in \mathcal{M}_R or in ${}_R\mathcal{M}$.*
- (vii) *R^2 has \perp equivalence.*

Proof. (v) \Rightarrow (i) and (v) \Rightarrow (vi) follow from Lemma 1 so we have the implications (v) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (vii) and (v) \Rightarrow (vi) \Rightarrow (vii)

(v) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (vii) is the left symmetric of (v) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (vii).

(vii) \Rightarrow (v) If R^2 has \perp equivalence, then by Lemma 3 we have that any 2 generated right (and any left) module has \perp equivalence, in particular R has orthogonal equivalence. Now let $R \xrightarrow{i} X$ be a monomorphism in \mathcal{M}_R such that $X/i(R)$ is 1 generated. Then as X has \perp equivalence, Proposition 7 shows that X is R cogenerated as right R module. Now by Proposition 9 i splits, as X is R cogenerated and R has \perp equivalence. Then we can apply Lemma 2 and obtain that R_R is injective. Because R has \perp equivalence, by Proposition 8 we obtain that R_R contains all isomorphism types of simple right modules, and as R_R is injective, we obtain that R_R is an injective cogenerator of \mathcal{M}_R , i.e. a right RF ring. Similarly we can show that R is also a left PF ring. \square

Corollary 2. *If R is a PF ring, then $F \simeq {}^*(F^*)$ by Φ_F for every finitely generated left module (the analogue holds for right modules).*

Proof. Proposition 6 shows that Φ_F is injective. By the same argument as in the proof of Lemma 1 we have that ${}_R R$ injective implies that Φ_F is an

epimorphism and the conclusion is proved. \square

Corollary 3. *R is a PF ring if and only if for every finitely generated right (or left) R module F , the lattice of the submodules of F is anti isomorphic to the lattice of the submodules of F^* via the \perp applications of Proposition 5, equivalently, the dual lattice of the submodules of any finitely generated right module is isomorphic (via \perp applications) to the lattice of the submodules of the dual of that module.*

As a consequence, we obtain two well known characterizations of (left and right) PF rings given by Kato [8].

Corollary 4. *The following assertions are equivalent:*

- (i) *R is a left and right PF ring.*
- (ii) *R is left and right cogenerator.*
- (iii) *Every factor module of R^2 (both in \mathcal{M}_R and ${}_R\mathcal{M}$) is torsionless.*

Proof. (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i) As ${}_R R$ is a cogenerator of ${}_R\mathcal{M}$, from Proposition 7 we get that $(X^\perp)^\perp = X$ for all left R modules M and submodules $X < M$ (by taking the class \mathcal{C} in Proposition 7 to be the category of left R modules). Using the fact that R_R is also a cogenerator (in \mathcal{M}_R), we get that $(I^\perp)^\perp = I$ for all right R modules N and submodules $I < N$; in particular we obtain that (the left and also the right R module) R^2 has orthogonal equivalence, and then Theorem 1 shows that R is a PF ring.

(ii) \Rightarrow (iii) If I is a right submodule of R^2 , then R^2/X is torsionless if and only if $(\{\bar{0}\}^\perp)^\perp = \{\bar{0}\}$ by Lemma 1 (where $\bar{0} \in R^2/X$ is the 0 element). Thus the hypothesis of (ii) is equivalent to the fact that R^2 has orthogonal equivalence, and the result follows again by Theorem 1. \square

Acknowledgement. This paper has been partially supported by a grant for young Ph.D. researchers by CNCSIS. The author wishes to thank his Ph.D. advisor prof. C. Năstăsescu - University of Bucharest for interesting and helpful remarks on the paper and for his continuous support.

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