THE ANTIPODE OF A DUAL QUASI-HOPF ALGEBRA WITH NONZERO INTEGRALS IS BIJECTIVE

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Dedicated to Fred Van Oystaeyen for his sixtieth birthday

Abstract. For A a Hopf algebra of arbitrary dimension over a field K, it is well-known that if A has nonzero integrals, or, in other words, if the coalgebra A is co-Frobenius, then the space of integrals is one-dimensional and the antipode of A is bijective. Bulacu and Caenepeel recently showed that if H is a dual quasi-Hopf algebra with nonzero integrals, then the space of integrals is one-dimensional, and the antipode is injective. In this short note we show that the antipode is bijective.

1. Introduction

The definition of quasi-Hopf algebras and the dual notion of dual quasi-Hopf algebras is motivated by quantum physics and dates back to work of Drinfel'd [4]. There has been recent interest in extending classical results and formulas to the quasi-Hopf setting (see [5] for example); the theory of integrals for quasi-Hopf algebras was studied in [10, 7, 2]. In [2], Bulacu and Caenepeel showed that a dual quasi-Hopf algebra is co-Frobenius as a coalgebra if and only if it has a nonzero integral. In this case, the space of integrals is one-dimensional and the antipode is injective, so that for finite dimensional dual quasi-Hopf algebras the antipode is bijective. In this note, we use the ideas from a new short proof of the bijectivity of the antipode for Hopf algebras by the second author [8] to show that the antipode of a dual quasi-Hopf algebra with integrals is bijective, thus extending the classical result of Radford [11] for Hopf algebras.

In this paper we prove

Theorem 1.1. *Let H be a co-Frobenius dual quasi-Hopf algebra, equivalently, a dual quasi-Hopf algebra having nonzero integrals. Then the antipode of H is bijective.*

2. Preliminaries

In this section we briefly review the definition of a dual quasi-Hopf algebra over a field K. We refer the reader to [1, 3, 12] for the basic definitions and properties of coalgebras and their comodules and of Hopf algebras. For the definition of dual quasi-Hopf algebra we follow [9, Section 2.4].

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Definition 2.1. *A dual quasi-bialgebra H over K is a coassociative coalgebra (H, Δ, ε) together with a unit $u : K \to H$, $u(1) = 1$, and a not necessarily associative multiplication $M : H \otimes H \to H$. The maps u and M are coalgebra maps. We write ab for $M(a \otimes b)$. As well, there is an element $\varphi \in (H \otimes H \otimes H)^*$ called the reassociator, which is invertible with respect to the convolution algebra structure of $(H \otimes H \otimes H)^*$. The following relations must hold for all $h, g, f, e \in H$:*

$$h_1(g_1f_1)\varphi(h_2, g_2, f_2) = \varphi(h_1, g_1, f_1)(h_2g_2)f_2 \tag{1}$$

$$1h = h1 = h \tag{2}$$

$$\varphi(h_1, g_1, f_1e_1)\varphi(h_2g_2, f_2, e_2) = \varphi(g_1, f_1, e_1)\varphi(h_1, g_2f_2, e_2)\varphi(h_2, g_3, f_3) \tag{3}$$

$$\varphi(h, 1, g) = \varepsilon(h)\varepsilon(g) \tag{4}$$

Here we use Sweedler's sigma notation with the summation symbol omitted.

Definition 2.2. *A dual quasi-bialgebra H is called a dual quasi-Hopf algebra if there exists an antimorphism S of the coalgebra H and elements $\alpha, \beta \in H^*$ such that for all $h \in H$:*

$$S(h_1)\alpha(h_2)h_3 = \alpha(h)1, \qquad h_1\beta(h_2)S(h_3) = \beta(h)1 \tag{5}$$

$$\varphi(h_1\beta(h_2), S(h_3), \alpha(h_4)h_5) = \varphi^{-1}(S(h_1), \alpha(h_2)h_3, \beta(h_4)S(h_5)) = \varepsilon(h). \tag{6}$$

Let H be a dual quasi-Hopf algebra. As in the Hopf algebra case, a left integral on H is an element $T \in H^*$ such that $h^*T = h^*(1)T$ for all $h^* \in H^*$; the space of left integrals is denoted by $\int_l$ and by [2, Proposition 4.7] has dimension 0 or 1. Right integrals are defined analogously with space of right integrals denoted by $\int_r$. Suppose $0 \neq T \in \int_l$. It is easily seen that $\int_l$ is a two sided ideal of the algebra H^*, and $KT \subseteq Rat(H^*)$ with right coaction given by $T \mapsto T \otimes 1$. Since for co-Frobenius coalgebras $Rat(H^*) = Rat({}_{H^*}H^*) = Rat(H^*_{H^*})$, KT must have left coaction $T \mapsto a \otimes T$. By coassociativity, a is a grouplike element, called the distinguished grouplike of H. Then, for all $h^* \in H^*$,

$$Th^* = h^*(a)T. \tag{7}$$

From [2, Proposition 4.2], the function $\theta^* : \int_l \otimes H \to Rat({}_{H^*}H^*)$

$$\theta^*(T \otimes h) = \sigma(S(h_5) \otimes \alpha(h_6)h_7) * (S(h_4) \rightharpoonup T) * \sigma^{-1}(S(h_3) \otimes \beta(S(h_2))S^2(h_1)) \tag{8}$$

is an isomorphism of right H-comodules, where $\sigma : H \otimes H \to H^*$ is defined by $\sigma(h \otimes g)(f) = \varphi(f, h, g)$, σ^{-1} is the convolution inverse of σ, and, as usual, $(h \rightharpoonup T)(g) = T(gh)$.

3. Proof of the theorem

Let H be a dual quasi-Hopf algebra with $0 \neq T \in \int_l$. As in [8], for each right H-comodule (M, ρ), we denote by ${}^aM$ the left H-comodule structure on M defined by $m^a_{-1} \otimes m^a_0 = aS(m_1) \otimes m_0$, where $\rho(m) = m_0 \otimes m_1$. Denote the induced right H^*-module structure on ${}^aM$ by $m \cdot^a h^* = h^*(m^a_{-1})m^a_0 = h^*(aS(m_1))m_0$. By [2, Corollary 4.4] the antipode S of H is injective, and therefore has a left inverse S^l. Then, for σ as above, we have the following analogue of [8, Proposition 2.5]:

Proposition 3.1. *The map $p : {}^aH \to Rat(H^*)$ defined by*

$$p(h) \quad = \quad \sigma(S(S^l(h_3)) \otimes \alpha(S^l(h_2))S^l(h_1)) * (h_4 \rightharpoonup T) * \sigma^{-1}(h_5\beta(h_6) \otimes S(h_7)) \tag{9}$$

is a surjective morphism of left H-comodules.

Proof. Let $\Psi(h) := \sigma(S(S^l(h_3)) \otimes \alpha(S^l(h_2))S^l(h_1))$. Then for $h^* \in H^*$ and $g \in H$:

$$\begin{array}{rcl}
(p(h) * h^*)(g) & = & p(h)(g_1)h^*(g_2) \\
(9) & = & \Psi(h_1)(g_1)T(g_2h_2)\sigma^{-1}(h_3\beta(h_4) \otimes S(h_5))(g_3)h^*(g_4) \\
& = & \Psi(h_1)(g_1)T(g_2h_2)\varphi^{-1}(g_3, h_3\beta(h_4), S(h_5))h^*(g_4) \\
& = & \Psi(h_1)(g_1)T(g_2h_2)\varphi^{-1}(g_3, h_3, S(h_5))h^*(g_4\beta(h_4)) \\
(5) & = & \Psi(h_1)(g_1)T(g_2h_2)\varphi^{-1}(g_3, h_3, S(h_7))h^*(g_4(h_4\beta(h_5)S(h_6))) \\
& = & \Psi(h_1)(g_1)T(g_2h_2)h^*(\varphi^{-1}(g_3, h_3, S(h_7))g_4(h_4\beta(h_5)S(h_6))) \\
& = & \Psi(h_1)(g_1)T(g_2h_2)\beta(h_5)h^*(\varphi^{-1}(g_3, h_3, S(h_7))g_4(h_4S(h_6))) \\
(1) & = & \Psi(h_1)(g_1)T(g_2h_2)\beta(h_5)h^*((g_3h_3)S(h_7)\varphi^{-1}(g_4, h_4, S(h_6))) \\
& = & \Psi(h_1)(g_1)T(g_2h_2)(S(h_7) \rightharpoonup h^*)(g_3h_3)\beta(h_5)\varphi^{-1}(g_4, h_4, S(h_6)) \\
& = & \Psi(h_1)(g_1)(T * (S(h_6) \rightharpoonup h^*))(g_2h_2)\beta(h_4)\varphi^{-1}(g_3, h_3, S(h_5)) \\
(7) & = & \Psi(h_1)(g_1)(S(h_6) \rightharpoonup h^*)(a)T(g_2h_2)\beta(h_4)\varphi^{-1}(g_3, h_3, S(h_5)) \\
& = & \Psi(h_1)(g_1)T(g_2h_2)\sigma^{-1}(h_3\beta(h_4) \otimes S(h_5))(g_3)h^*(aS(h_6)) \\
(9) & = & p(h_1)(g)h^*(aS(h_2)) \\
& = & p(h^*(aS(h_2))h_1)(g) \\
& = & p(h^*(h^a_{-1})h^a_0)(g) \\
& = & p(h \cdot^a h^*)(g).
\end{array}$$

Thus p is left H-colinear. Finally, we note that $p \circ S = \theta^*(T \otimes -)$ where θ^* is the isomorphism from (8) so that p is surjective. □

Remark 3.2. *As the referee pointed out, the computation in the proof of Proposition 3.1 may be streamlined using a formula that generalizes the equality $h_1 \otimes h_2S(h_3) = h \otimes 1$ which holds for any h in a Hopf algebra H. If H is a dual quasi-Hopf algebra, and $p_R \in (H \otimes H)^*$ is given by $p_R(h, g) = \varphi^{-1}(g, h_1\beta(h_2), S(h_3))$, we then have*

$$p_R(g_1, h)g_2 = p_R(g_2, h_2)(g_1h_1)S(h_3)$$

for all $h, g \in H$. As a consequence we get that

$$T(g_1h_1)p_R(g_2, h_2)g_3 = T(g_1h_1)p_R(g_2, h_2)aS(h_3)$$

for all $h, g \in H$, and this formula allows us to delete lines four through eleven of the above mentioned computation.

Let c be a grouplike element of H. From [2, p.580], c is invertible with inverse $S(c)$. We will show that left multiplication by c has an inverse too.

Let $\theta_c \in \mathrm{End}(H)$ be defined by $\theta_c(h) = ch$ and define the coinner automorphisms q_c and $r_c = q_c^{-1} \in \mathrm{End}(H)$ by:

$$q_c(h) = \varphi^{-1}(c, S(c), h_1)h_2\varphi(c, S(c), h_3) \text{ and } r_c(h) = \varphi(c, S(c), h_1)h_2\varphi^{-1}(c, S(c), h_3).$$

Lemma 3.3. *For any grouplike element c and θ_c, r_c, q_c as above, $\theta_c \circ \theta_{c^{-1}} = r_c$ and thus θ_c is bijective with inverse $\theta_c^{-1} = \theta_{c^{-1}} \circ q_c = q_{c^{-1}} \circ \theta_{c^{-1}}$.*

Proof. Using (1) and the fact that $c^{-1} = S(c)$, we see that

$$\theta_c \circ \theta_{c^{-1}}(h) = c(c^{-1}h) = \varphi(c, S(c), h_1)(cS(c))h_2\varphi^{-1}(c, S(c), h_3) = r_c(h).$$

The same formula for $c^{-1} = S(c)$ yields $\theta_{c^{-1}} \circ \theta_c = r_{c^{-1}}$ and the statement then follows directly. □

We can now prove our main result.

Proof of Theorem 1.1.
We only need to prove the surjectivity. The proof goes along the lines of the proof of [8, Theorem 2.6], but with the difference that here the antipode is not necessarily an anti-morphism of algebras.

Let π be the composition map ${}^aH \xrightarrow{p} Rat(H^*_{H^*}) \xrightarrow{\sim} H \otimes \int_r \simeq H$, where the last two isomorphisms follow by left-right symmetry of the results of [2]. Since H is a co-Frobenius coalgebra, ${}^HH$ is projective by [6, Theorem 1.3] or [2, Theorem 4.5, (x)], and as π is surjective, there is a morphism of left H-comodules $\lambda : H \to {}^aH$ such that $\pi\lambda = \mathrm{Id}_H$. We then have

$$aS(\lambda(h)_2) \otimes \lambda(h)_1 = \lambda(h)^a_{-1} \otimes \lambda(h)^a_0 = h_1 \otimes \lambda(h_2).$$

Applying $\mathrm{Id} \otimes \varepsilon\pi$, we get $aS(\varepsilon\pi(\lambda(h)_1)\lambda(h)_2) = h$ for any $h \in H$. Thus $\theta_a \circ S$ is surjective and since θ_a is bijective by Lemma 3.3, S is surjective also. □

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References

[1] E. Abe, *Hopf Algebras*, Cambridge Univ. Press, 1977.
[2] D. Bulacu, S. Caenepeel, Integrals for (dual) quasi-Hopf algebras. Applications, J. Algebra **266** (2003), no. 2, 552-583.
[3] S. Dăscălescu, C. Năstăsescu, Ş. Raianu, *Hopf Algebras: an Introduction*, Monographs and Textbooks in Pure and Applied Mathematics **235**, Marcel Dekker, Inc., New York, 2001.
[4] V.G. Drinfel'd, Quasi-Hopf Algebras, Leningrad Math. J. **1** (1990) 1419-1457.
[5] W. Ferrer Santos, I. Lopez Franco, Monoidal categories of comodules for coquasi Hopf algebras and Radford's formula, arXiv:0801.1133v1 [math.QA].
[6] J. Gomez-Torrecillas, C. Năstăsescu, Quasi-co-Frobenius coalgebras, J. Algebra **174** (1995), no. 3, 909-923.
[7] F.Hausser, F. Nill, Integral theory for quasi-Hopf algebras, arXiv:math/9904164v2.
[8] M.C. Iovanov, Generalized Frobenius algebras and the theory of Hopf algebras, preprint, arXiv:0803.0775.
[9] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge, 1995.
[10] F. Panaite, F. Van Oystaeyen, Existence of integrals for finite dimensional Hopf algebras, Bull. Belg. Math. Soc. Simon Stevin **7** (2000) 261-264.

[11] D.E. Radford, Finiteness conditions for a Hopf algebra with a nonzero integral, J. Algebra **46** (1977), no. 1, 189-195.
[12] M.E. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.

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