GRASSMANNIAN SEMIGROUPS AND THEIR REPRESENTATIONS

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ABSTRACT. The set of row reduced matrices (and of echelon form matrices), is closed under multiplication. We show any system of representatives for the $Gl_n(\mathbb{K})$ action on the $n \times n$ matrices, which is closed under multiplication, is necessarily conjugate to one that is in simultaneous echelon form. We call such closed representative systems Grassmannian semigroups. We study internal properties of such Grassmannian semigroups, show that they are algebraic semigroups and admit gradings by the finite semigroup of partial permutations, with components that are naturally in one-one correspondence with the Schubert cells of the total Grassmannians. We show that there are infinitely many isomorphism types of such semigroups in general, and two such semigroups are isomorphic exactly when they are semiconjugate. We also investigate their representation theory over an arbitrary field, and other connections with multiplicative structures on Grassmannians and Young diagrams.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{K}$ be an infinite field. Consider the left regular action of the general linear group $Gl_n(\mathbb{K})$ on the matrices $M_n(\mathbb{K})$. A very important set of matrices is the set $\mathcal{R}$ of row reduced matrices, which is a standard system of representatives for this action. Row reduction is the basic algorithm for solving linear systems. Moreover, $\mathcal{R}$ has an additional remarkable property: it is closed under the multiplication of matrices. Strangely, this basic linear algebra fact is not as well known as one would expect; we have only noted this fact in one textbook [8], Exercise 2.19; it is also noted in [9], page 67 (see also [1]). The fact that a product of row reduced matrices is row reduced has a geometric consequence. Consider V the n-dimensional space over $\mathbb{K}$, with a fixed basis $\{e_1, \ldots, e_n\}$, regarded as the vector space of column vectors with n entries, and $M_n(\mathbb{K})$ acting as endomorphisms on the left. Then each $Gl_n(\mathbb{K})$ orbit $\mathcal{O}$ corresponds uniquely to a subspace of $\mathbb{K}^n$, by $\mathcal{O} = Gl_n(\mathbb{K}) \cdot A \to \ker(A)$ (since $\ker(A)$ is uniquely determined for $A \in \mathcal{O}$). Hence, the multiplication of row reduced matrices induces a multiplication on the total Grassmannian $Gr_{\mathbb{K}}(n)$ via the bijection $\mathcal{R} \longleftrightarrow Gr_{\mathbb{K}}(n)$, $A \longleftrightarrow \ker(A)$. Thus $\mathcal{R}$ and $Gr_{\mathbb{K}}(n)$ form a semigroup, and in fact, an algebraic semigroup. This motivates the introduction of the following terminology.

Definition 1.1. *We say S is a Grassmannian semigroup if S is a system of representatives for the left regular $Gl_n(\mathbb{K})$ action on $M_n(\mathbb{K})$, which is also closed under multiplication of matrices. Equivalently, $\mathcal{S}$ is a sub-semigroup of $(M_n(\mathbb{K}), \cdot)$ such that for every subspace W of V, there is a unique element $A \in S$ such that $\ker(A) = W$.*

Grassmannian semigroups were also studied in [1]. It is an interesting question to ask what other special systems of representatives for the $Gl_n(\mathbb{K})$ action can be found, or equivalently, other natural multiplicative structures on $Gr_{\mathbb{K}}(n)$ compatible with the natural matrix multiplication, and what is special about $\mathcal{R}$, and whether $\mathcal{R}$ somehow canonical. This is also interesting from the perspective of row reduction: what are the interesting canonical forms for row reduction?

Our first main result addresses this question. In Proposition 2.9 and Theorem 2.6 we show that every such Grassmannian semigroup is conjugate to a Grassmannian semigroup which is in simultaneous echelon form with all initial pivots equal to 1 (i.e. a Grassmannian semigroup consisting only of row echelon form matrices), but where entries above pivots are not necessarily equal to 0. This is a consequence of a peculiarity of such semigroups, namely, that if two elements in a Grassmannian semigroup have the same rank, then they must have the same range (column space). Beyond being conjugate to a semigroup in echelon form, we prove there is a certain canonical form for each such semigroup, which generalizes the set of row reduced matrices. In particular, we are able to describe all such Grassmannian semigroups of order 2 and 3.

For each row echelon form matrix, we can define its type to be the set of positions of columns containing pivots. The set of row reduced matrices with only 0's everywhere except pivot positions form a semigroup Π_n of 2^n elements, and any Grassmannian semigroup S is graded by Π_n. This allows one to obtain a natural one-to-one correspondence between Schubert cells of the total Grassmannian $Gr_{\mathbb{K}}(n)$, associated Young diagrams and the graded components of such a semigroup. In particular, for example, the semigroup of row reduced matrices and $Gr_{\mathbb{K}}(n)$ have an algebraic semigroup structure.

In Section 3, we study the algebraic structure of Grassmannian semigroups S. We show that there are several elements in such an algebraic structure that can be identified and defined intrinsically. There is a basis B in which S is echelon, and when S is in echelon form, the Jordan cell N of dimension n and eigenvalue 0 must belong to the semigroup S; also, the rank k row reduced diagonal idempotents E_k must be in the (echelon form of the) semigroup. This element N is uniquely determined intrinsically in S by the algebraic fact that every nilpotent element of S is a left multiple of N. Moreover, the equivalence relation on the set $\mathcal{E}$ of idempotents in S defined by $E \sim E'$ if $EE' = E'$ and $E'E = E$ partitions $\mathcal{E}$ into $n+1$ equivalence classes, and allows one to define the rank, and also the type of an element of S independently of the ambient algebra $M_n(\mathbb{K})$ in which S is defined. Hence, given a Grassmannian semigroup S, all this structure, including the partition (grading) of S into parts indexed by Π_n can be recovered from internal algebraic properties of S. This indicates that the algebraic structure (multiplication) of this S may retain plenty of the combinatorics and geometry of $Gr_{\mathbb{K}}(n)$.

In Section 4, we study the problem of isomorphisms of Grassmannian semigroups. One sees easily that isomorphisms of such algebraic structures determine inclusion preserving bijections on $Gr_{\mathbb{K}}(n)$, so that the basic Fundamental Theorem of Projective Geometry can be used. Our main result of this section is that two Grassmannian semigroups are isomorphic if and only if they are semiconjugate, that is, they are isomorphic via a *ring* automorphism of $M_n(\mathbb{K})$, which must be the composition of a conjugation by a matrix A and a ring automorphism $\overline{\sigma}$ of $M_n(\mathbb{K})$ induced by some $\sigma \in \text{Aut}(\mathbb{K})$. This, together with the results of the first section allows us to determine up to isomorphism all such algebraic Grassmannian semigroups of dimensions 2 and 3, and determine the cardinality of the set of all such isomorphism classes. For example, when $\mathbb{K} = \mathbb{R}$, we show that there are $\aleph_2 = 2^{2^{\aleph_0}}$ such isomorphism classes. This can be done in higher dimensions, but the descriptions one would obtain makes such results impractical to state.

In Section 5, we further explore connections between the Grassmannian semigroups, Grassmannians and Young diagrams. We note that besides the multiplicative structures they induce on $Gr_{\mathbb{K}}(n)$, one obtains a monoid structure on the set of Young diagrams. Each matrix in Π_n represents a Schubert cell in $G_{\mathbb{K}}(n)$, and has a canonically associated Young diagram, and vice-versa, and one can define a bijective function between the set of *all* Young diagrams and the monoid

(semigroup) $\Pi = \bigcup_{n=1}^{\infty} \Pi_n$, where Π_n is regarded naturally as a submonoid of Π_{n+1} as corner matrices. This is interesting vis-a-vis the so called plactic monoid, a monoid structure on the set of *all* Young tableaux. Motivated also by this, we study the representation theory of Grassmannian semigroups, and of the monoids Π_n and Π. The semigroup algebra $\mathbb{F}[S]$ over some arbitrary possibly different field $\mathbb{F}$ of a Grassmannian semigroup S on $M_n(\mathbb{K})$ is in fact semilocal and has nilpotent Jacobson radical (although S can be quite large). We also completely determine the Ext quiver of Π_n, its left and right projective indecomposables and their dimensions; these are expressed in terms of the combinatorial binomial coefficients.

We note that in our treatment and interpretation of $Gr_{\mathbb{K}}(n)$ we take the naive approach and view it as simply a union of affine spaces as opposed to the usual projective space subvariety via the Plücker embedding; as noted before, this has the advantage of being compatible with other algebraic structures (such as the natural matrix multiplication). In algebraic terms, this translates into the difference between our semigroup algebra $\mathbb{F}[\Pi_n]$ and the exterior algebra $\Lambda(\mathbb{F}^n)$. Nevertheless, some questions regarding these structures arise on the way, such as whether there are other relevant connections of the above mentioned product of Young diagrams with other combinatorial representation theory or algebraic combinatorics problems. We believe another interesting question is to completely describe the semigroup algebras of Π_n as quivers with relations, and determine the properties of the semigroup algebra $\mathbb{F}[\Pi]$ of the semigroup of Young diagrams. They are also bialgebras, and determining their representation and Grothendieck rings could be an interesting problem as well. The algebraic semigroup structures we find on Grassmannian semigroups also naturally give rise dually to bialgebras (the bialgebra of algebraic representative functions). Hence, we hope this work can be the starting point of future investigations.

2. Simultaneous echelon form for Grassmannian semigroups

The following is an easy observation that is likely known; we include a brief argument for completeness.

Proposition 2.1. *Let V be an n-dimensional vector space over an infinite field $\mathbb{K}$ and $X = \{A_i \mid i = 0, 1, \ldots, k\}$ be a finite collection of subspaces of V. Then there is a flag $0 = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_n = V$ on V with $\dim(B_i) = i$ such that if $\dim(A_j) = t$ then B_{n-t} is a complement for A_t.*

Proof. Fix a basis $e_1, \ldots, e_n$ and write all vectors as column vectors with respect to this basis. Let $B = [x_{ij}]$ be a generic $n \times n$ matrix with variables as shown. Let $\dim A_i = d_i$. Replace the first d_i columns of B with a basis for A_i and denote it by $M(A_i)$. Let $\Psi(A_i) = \det M(A_i)$, so $\Psi(A_i)$ is a polynomial in $\mathbb{K}[x_{ij} | i, j]$ (depending only on $n(n - d_i)$ variables). Let $\Psi = \prod \Psi(A_i)$, so Ψ is a polynomial in the n^2 variables x_{ij}. Note that Ψ is the zero polynomial only when $\det M(A_i) = 0$ for some i. But this is not possible since $M(A_i)$ can always be completed to a basis (so A_i has a complement), and $\det(M(A_i))$ would be non-zero at the corresponding point. Therefore, since the field $\mathbb{K}$ is infinite, Ψ has a non-zero $B \in M_n(\mathbb{K})$. If B_i is be the span of the last $n - i$ columns of B and it is clear that the B_i are the members of C(X) above required. $\square$

Common complements have been studied for a long time, and in the generality of modules over arbitrary rings, most recently by T.Y. Lam and his co-authors; we refer to [2, 5, 7] also for the history of the subject. This is usually studied in the form of existence of common complements of isomorphic submodules; for vector spaces, a much more general statement is possible as above.

In what follows, S will be a Grassmannian semigroup of $n \times n$ matrices.

Proposition 2.2. *Let $A \in S$ be a matrix of rank k. Then for every subspace W of V such that $V = W \oplus \operatorname{Im}(A)$, let $E \in S$ be the unique element such that $\ker(E) = W$. Then E is an idempotent and $EA = A$. Consequently, the column space of A and E coincide (i.e. $\operatorname{Im}(E) = \operatorname{Im}(A)$) and the columns of A span the 1-eigenspace of E.*

Proof. Let $E \in S$ be such that $\ker(E) = W$ (this is obviously unique). Note that since $W \cap \operatorname{Im}(A) = 0$, we have that $\dim(\operatorname{Im}(EA)) = \dim(E(\operatorname{Im}(A))) = \dim(E(W + \operatorname{Im}(A))) = \dim(\operatorname{Im}(E))$ so $\operatorname{rank}(EA) = \operatorname{rank}(E) = n - \dim(W) = \operatorname{rank}(A)$. This means that $\dim(\ker(A)) = \dim(\ker(EA))$, and since $\ker(A) \subseteq \ker(EA)$, it follows that $\ker(A) = \ker(EA)$. By the uniqueness property of the Grassmannian semigroup, it follows that $EA = A$.
Now note that $E^2 A = EA = A$, and as above, this shows that $\operatorname{Im}(E^2) = \operatorname{Im}(E)$, hence $\ker(E^2) = \ker(E)$. Therefore, $E^2 = E$ again by the uniqueness property. Moreover, the identity $EA = A$ shows that the columns of A are 1-eigenvectors for E, and since $\operatorname{Im}(E)$ has dimension k and is spanned by eigenvectors, it follows that $\operatorname{Im}(A) \subseteq \operatorname{Im}(E)$, so they coincide since they have the same dimension (the equality $EA = A$ shows this directly also). $\square$

Proposition 2.3. *Let $A, B \in S$ such that $\operatorname{rank}(A) = \operatorname{rank}(B)$. Then $\operatorname{Im}(A) = \operatorname{Im}(B)$.*

Proof. Let k be the common rank of A and B and assume $\operatorname{Im}(A) \neq \operatorname{Im}(B)$. Note that there is a subspace W of V such that $V = \operatorname{Im}(A) \oplus W = \operatorname{Im}(B) \oplus W$. This can be seen by the remarks in the beginning of this section, or directly: take A', respectively B', to be $n \times k$ matrices formed by some vectors that span $\operatorname{Im}(A)$, respectively $\operatorname{Im}(B)$. finding the W amounts to finding an $n \times (n - k)$ matrix U such that $\det[A'|U] \neq 0$ and $\det[B'|U] \neq 0$. This is possible, since the sets $\{U, \det[A'|U] \neq 0\}$ and $\{U, \det[B'|U] \neq 0\}$ are open subsets of the affine space $\mathbb{A}^{n(n-k)}$.
By Proposition 2.2, there are idempotents $E, F \in S$ such that $\ker(E) = \ker(F) = W$ and $\operatorname{Im}(E) = \operatorname{Im}(A)$, $\operatorname{Im}(F) = \operatorname{Im}(B)$. This shows that $E \neq F$, and this contradicts the Grassmannian semigroup property, since two elements in S have the same kernel. $\square$

We can now note the following interesting fact about the elements of a Grassmannian semigroup.

Corollary 2.4. *There are subspaces $V_0, V_1, \ldots, V_n$ with $\dim(V_k) = k$ and such that*
(i) If $A \in S$ has $\operatorname{rank}(A) = k$, then $\operatorname{Im}(A) = V_k$.
(ii) For each k there are idempotents $E \in S$ such that $\operatorname{Im}(E) = V_k$.

Proposition 2.5. *With the notations of the previous proposition, for each $0 \leq k < n$, $V_k \subset V_{k+1}$.*

Proof. Let E_k be an idempotent with $\operatorname{Im}(E_k) = V_k$. If $W_k = \ker(E_k)$, obviously $\ker(E_k) \cap \operatorname{Im}(E_k) = 0$ since E_k is an idempotent. Let W be a subspace of codimension 1 in W_k; note that it exists since $W_k \neq 0$ since $k < n$. Let $A \in S$ be such that $\ker(A) = W$. We note that since $\operatorname{Im}(E_k) \cap \ker(A) = 0$, we have $A(\operatorname{Im}(E_k)) \cong \operatorname{Im}(E_k)$, so $\operatorname{rank}(AE_k) = \operatorname{rank}(E_k)$. As before, since $\ker(E_k) \subseteq \ker(AE_k)$, using the uniqueness property of S, this shows that $AE_k = E_k$. From this we obtain that $\operatorname{Im}(E_k) \subset \operatorname{Im}(A)$. But $\operatorname{rank}(A) = n - \dim(W) = k + 1$, so $\operatorname{Im}(A) = V_{k+1}$ and $\operatorname{Im}(E_k) = V_k$. Thus, $V_k \subset V_{k+1}$. $\square$

Hence, for every element $A \in S$, we have $A(V_k) = V_{k-i}$ for some i. Using this for the flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$, and choosing a basis $(w_i)_{i=1,\ldots,n}$ such that $w_i \in V_i \setminus V_{i-1}$, we see that with respect to this basis every endomorphism $A \in S$ is in echelon form (here by echelon form we understand the usual, that is, a matrix for which in every row the first non-zero element is found at least one position to the right from the first non-zero element in the previous row).

Thus, we have the following

Theorem 2.6. *Any Grassmannian semigroup of $n \times n$ matrices may be conjugated to one which is in echelon form (that is, to a semigroup consisting of matrices in echelon form; therefore, the matrices of a Grassmannian semigroup are simultaneously echelonizable).*

Let us denote E_k the "basic" idempotent matrices $E_k = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & & & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$, having

1 on the first k entries on the diagonal and 0 elsewhere. The following is a variant of a result present in [1]; we give there a more direct short proof.

Proposition 2.7. *Let $\mathcal{S}$ be a Grassmannian semigroup in $M_n(\mathbb{K})$, which is in echelon form. Then $E_k \in \mathcal{S}$ for all $1 \leq k \leq n$.*

Proof. Let $F_k \in \mathcal{S}$ be the unique matrix whose kernel is $e_{k+1}, \dots, e_n$, where e_i are the vectors of the canonical basis. As in Proposition 2.2, F_k is an idempotent, and since F_k are in echelon form, we have $F_k = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, where A is a $k \times k$ upper triangular matrix. Since A is diagonalizable $(A^2 = A)$ with $\text{rank} A = k$, we see that $A = I_k$, the $k \times k$ identity, and so $F_k = E_k$, and the proof is finished. $\square$

We also note a more general easy description of the idempotents in a Grassmannian semigroup.

Proposition 2.8. *Let $\mathcal{S}$ be a Grassmannian semigroup in echelon form. Then the idempotents of rank k in $\mathcal{S}$ are of the form*

$$E = \begin{pmatrix} I_k & C \\ 0 & 0_{n-k} \end{pmatrix}$$

where C is an arbitrary $k \times (n - k)$ matrix with entries in $\mathbb{K}$.

Proof. Since E has rank k and is in echelon form, in particular $E = \begin{pmatrix} H & C \\ 0 & 0_{n-k} \end{pmatrix}$. Since $E^2 = E$, we get $H^2 = H$ and $HC = C$. This shows that the columns of C are 1-eigenvectors for the idempotent H, and so they are linear combinations of the columns of H. Hence, $\text{rank}(E) = \text{rank}(H) = k$, and since H is an idempotent it follows that $H = I_k$, and so

$$E = \begin{pmatrix} I_k & C \\ 0 & 0_{n-k} \end{pmatrix}$$

Conversely, if C is an arbitrary $k \times (n - k)$ matrix, note that the span of the columns of the $n \times (n - k)$ matrix $M(C) = \begin{pmatrix} -C \\ I_{n-k} \end{pmatrix}$ is a subspace $V(C)$ of $V = \mathbb{K}^n$ of dimension $n - k$ for which $V(C) \cap V_k = 0$. Moreover, for every such subspace W for which $W \cap V_k = 0$ there is a unique such matrix C for which $W = V(C)$. Indeed, let B be an $k \times (n - k)$ matrix whose columns span W, and column reduce this matrix. Note that a lines $1, 2, \dots, k$ cannot contain a pivot, since $V_k \cap W = 0$, and the conclusion follows as $M(C)$ and $M(D)$ are not column equivalent if $C \neq D$. Finally, for every $k \times (n - k)$-matrix C there is an element $E \in \mathcal{S}$ with $\ker(E) = V(C)$, and since $V(C) \cap V_k = 0$, it follows that E is an idempotent by Proposition 2.2. Hence $E = \begin{pmatrix} I_k & D \\ 0 & 0_{n-k} \end{pmatrix}$ and E annihilates $V(C)$, so $EM(C) = 0$ from which it follows that $D = C$. $\square$

Proposition 2.9. *Assume $\mathcal{S}$ is a Grassmannian semigroup in echelon form with respect to a basis $e_1, \ldots, e_n$. Then, after a change of basis of the type $e_i \mapsto \lambda_i e_i$ (i.e. a "diagonal" change of basis) the matrices in S will have 1 on all pivot entries.*

Proof. First, we show that all rank 1 matrices with a pivot on position $(1, i)$ will have the same value of the pivot. If $A_i \in \mathcal{S}$ is matrix having a pivot value of a_i at position $(1, i)$,

$$A = \begin{pmatrix} 0 & \ldots & a_i & a_{i+1} & \ldots & a_n \\ 0 & 0 & & \ldots & & 0 \\ \ldots & & \ldots & & \ldots & \\ 0 & 0 & & \ldots & & 0 \end{pmatrix}, \text{ then } B_i = A_i \cdot E_i \in \mathcal{S} \text{ has the element } a_i \in \mathbb{K} \text{ as a}$$

pivot in position $(1, i)$ and 0 elsewhere. By the uniqueness condition for Grassmannian semigroups, such an element in $\mathcal{S}$ is unique, so a_i is the same for all matrices of this form.

Now, consider a matrix $B \in \mathcal{S}$, with a pivot in position (i, j) equal to b_{ij}. Then it is straightforward to note that $B_i \cdot B$ is an echelon matrix in $\mathcal{S}$ having a pivot of $a_i b_{ij}$ in position $(1, j)$, and so, by the above considerations we see that $a_j = a_i b_{ij}$. We now note that if we change bases by $e_i' = \frac{1}{a_i} e_i$, in the new basis e_i', each element of $\mathcal{S}$ will still be in echelon form, and the rank 1 matrices will have the pivots equal to 1. Moreover, by the above, the pivots in positions (i, j) will be $b_{ij} = a_j \cdot a_i^{-1} = 1$. $\qquad\square$

It seems appropriate here to note the following remark on the structure on another set of elements which occur in every Grassmannian semigroup which in fact form the set nilpotents of rank 1.

Remark 2.10 (Elements of type (p), $1 \le p \le n$). *If $\mathcal{S}$ is a Grassmannian semigroup in echelon form, then for every $1 \le p \le n$, the elements of type (p) in $\mathcal{S}$ are all the matrices $L_p(a_{p+1}, \ldots, a_n)$ for arbitrary $a_{p+1}, \ldots, a_n$, having the first line $(0, \ldots, 0, 1, a_{p+1}, \ldots, a_n)$, with the 1 on position $(1, p)$ and 0 elsewhere. Indeed, for arbitrary $a_{p+1}, \ldots, a_n$ in $\mathbb{K}$, there has to be an element $B \in \mathcal{S}$ of rank 1 which has kernel the subspace $x_p = -a_{p+1}x_{p+1} - \cdots - a_n x_n$ (such a subspace is uniquely determined by $a_{p+1}, dots, a_n$), and in row reduced form the matrix $B = L_p(a_{p+1}, \ldots, a_n)$ is the only such possibility. Obviously, these are precisely the nilpotent elements of rank 1.*

2.1. **A "row reduced" form for Grassmannian semigroups.** We give a theorem which shows that, after conjugation by a suitable element, the matrices in a Grassmannian semigroup $\mathcal{S}$ can be put (simultaneously) in a form very close to the classical row reduced echelon form. First, we fix some notation.

Definition 2.11. *Let $\tau = (k_1, k_2, \ldots, k_t)$ be a t-uplet of integers where $1 \le k_1 < k_2 < \cdots < k_t \le n$ are integers, $1 \le t \le n$. We will say an echelon matrix A has shape τ if it has pivots at positions (i, k_i), for $i = 1, \ldots, t$, i.e. the pivots are at columns $k_1, \ldots, k_t$.*

We denote by P_τ the matrix having 1 at positions (i, k_i) and 0 elsewhere. In what follows, we it will be convenient to consider column reduced matrices, which mean that we column-reduce right to left, bottom-up.

Definition 2.12. *We say that a matrix N is right column reduced (respectively, in right column echelon form) if it is obtained from a row reduced matrix (respectively, from an echelon matrix) via reflection across the secondary diagonal; equivalently, if the matrix under consideration is rotated ninety degrees clockwise and reflected across a vertical line to its left, it is row reduced.*

That is, a matrix N is right column reduced if its columns, listed from right to left are $c_1, c_2, \ldots, c_n$, and these columns, when transposed and organized into lines in reverse order

as ${}^t c_n, \ldots, {}^t c_1$, form a row reduced matrix that we will denote by $R(N)$. Obviously, this operation R is its own inverse, so $R^2 = Id$ on $M_n(\mathbb{K})$. Its importance is revealed when dealing with the nullspace of matrices, and offers a convenient way to write such nullspaces.

For a shape $\tau = (k_1, \ldots, k_t)$, let τ' be the shape defined as $\tau' = (l_1, \ldots, l_s)$, such that $1 \leq l_1 < \cdots < l_s \leq n$ and $\{l_1, \ldots, l_s\} \sqcup \{k_1, \ldots, k_t\} = \{1, 2, \ldots, n\}$ (i.e. $\{l_1, \ldots, l_s\}$ is the complement of $\{k_1, \ldots, k_t\}$). The correspondence between τ and τ' can also be explained via conjugate (transpose) Young diagrams.

We recall a standard construction done for Schubert cells of Grassmannians. If A is a row echelon (or row reduced) matrix of type τ, let $Y_0(A)$ be the Young diagram obtained by retaining all non-pivot positions from all rows containing pivots. Namely, if A has type $\tau = (k_1, \ldots, k_t)$, we place in row i of $Y_0(\tau)$ a number of boxes equal to the number of non-pivot positions on row i, to the right of (i, k_i). More precisely, if A has a pivot at (i, k_i) there are $n - k_i$ columns to the right of column k_i, $t - i$ of which are pivot columns. There are therefore $(n - k_i) - (t - i) = n - t + i - k_i$ non pivot positions on row i and to the right of column k_i. Hence, we may write $Y_0(\tau) = (n - t + 1 - k_1, n - t + 2 - k_2, \ldots n - k_t)$, where the j'th entry of this array denotes the number of boxes of $Y_0(\tau)$ on row j. This is using the French convention with rows having a non-increasing number of boxes as we go downwards. If $\tau' = (l_1, \ldots, l_s)$, let $Y_0(\tau')$ be defined similarly. Then it is not difficult to see that $Y_0(\tau)$ and $Y_0(\tau')$ are conjugate Young diagrams. We will later use further connections of row reduced and echelon form matrices and Young diagrams.

We now extend the map R to a bijection from echelon matrices of type τ to right column echelon matrices of type τ'. It will also be useful to have some additional notation. If $\tau = (k_1, \ldots, k_t)$ is a fixed shape, let us denote by W_τ the set of matrices which entries 0 everywhere except possibly non-zero at positions (i, j) for $i \leq t$ and $j > k_i$, and $j \notin \{k_1, \ldots, k_t\}$. Equivalently, $A \in W_\tau$ if and only if $A + P_\tau$ is a row reduced matrix of type τ. Obviously, W_τ is a $\mathbb{K}$-subspace of $M_n(\mathbb{K})$. Note that the dimension of this space is $\dim(W_\tau) = (n - k_1 - t + 1) + (n - t - k_2 + 2) + \cdots + (n - t - k_t + t) = t(n-t) + t(t+1)/2 - (k_1 + \cdots + k_t)$. We will more closely investigate the relation of Grassmannian semigroups and Grassmannians later.

Remark 2.13. *A t-shuffle is a permutation on n letters that preserves the order of $(1, \ldots, t)$ and $(t + 1, \ldots, n)$. If A is a matrix and P is a permutation matrix, then AP permutes the columns of A. It is easy to see that a shuffle is a permutation matrix of the form $P = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$ where R_1 and R_2 are 0,1 echelon matrices. Every rank t row reduced matrix is of the form $R = \begin{pmatrix} I_t & N \\ 0 & 0 \end{pmatrix} P$. Let $M = \begin{pmatrix} I_t & N \\ 0 & 0 \end{pmatrix}$; then $\ker(M)$ is the span of the columns of $\begin{pmatrix} 0 & -N \\ 0 & I_{n-t} \end{pmatrix}$. So, the kernel (null space) of R is the span of the columns of $N(R) := P^{-1} \begin{pmatrix} 0 & -N \\ 0 & I_{n-t} \end{pmatrix} = \begin{pmatrix} R_1^t & R_2^t \end{pmatrix} \begin{pmatrix} 0 & -N \\ 0 & I_{n-t} \end{pmatrix}$. But the shape of R is the shape of R_1 and one sees that the shape of S, according to our right-left, down-up convention is the shape of R_2^t, which is the shape of the complement of R_1 as desired.*

We have thus defined, for each row reduced matrix R as above, a right column reduced matrix $N(R)$ such that the null space of R is the span of the columns of $N(R)$; furthermore, if R has type $\tau = (k_1, \ldots, k_t)$, then $N(R)$ is a right column reduced matrix of type $\tau' = (l_1, \ldots, l_s)$ as follows with τ' being the complement of τ. Below we see an example on how this construction

N works.

$$R = \begin{pmatrix} 1 & a & 0 & 0 & b & c \\ 0 & 0 & 1 & 0 & d & e \\ 0 & 0 & 0 & 1 & f & g \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow N(R) = \begin{pmatrix} 0 & 0 & 0 & -a & -b & -c \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d & -e \\ 0 & 0 & 0 & 0 & -f & -g \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We denote by S the map that does the reverse operation, so for a right column reduced matrix B of type τ', it associates a row reduced matrix $S(B)$ of type τ. We have that S and N are inverse maps. We summarize the properties of these in the following Lemma, which is likely known, and only amounts to a careful computational observation, and therefore we omit the details of the proof.

Lemma 2.14. *(i) Let B be a right column reduced matrix of type τ'. Then there is a unique matrix A of type τ, such that $AB = 0$. Moreover, $A = S(B)$.*
(ii) Let A be a row reduced matrix of type τ. Then there is a unique right column reduced matrix B of type τ', such that $AB = 0$. Moreover, $B = N(A)$.
(iii) For A, B as in (i) and (ii) above, we have $BA = 0$.

We have the following theorem representing matrices A of type τ which are in row reduced form, and which have the null space equal to N of type τ'. If we regard $\mathbb{K}^n$ as the space of column vectors over $\mathbb{K}$, we first note that every subspace W of $\mathbb{K}^n$ as a canonical basis which can be represented uniquely by a matrix B in right column reduced form. This is obtained by column reducing an arbitrary basis of W.

Theorem 2.15. *Let W be a subspace of $\mathbb{K}^n$. Let N be the right column reduced matrix whose columns represent a basis of W, and let τ' be the type of N. If S is an echelon matrix of type τ such that $SN = 0$, then there is a matrix C which has 0 entries everywhere except potentially at positions above the pivot positions of S, and such that $S = S(N) + C - CN$.*

We saw Grassmannian semigroups can be put into (simultaneous) echelon form. Using the previous theorem, we notice a structure statement for matrices in a Grassmannian semigroup, which will show an even closer resemblance to the the semigroup of row reduced matrices. Let $\mathcal{S}$ be in echelon form with the pivots of every element of $\mathcal{S}$ equal to 1. For each type τ, and every matrix N of type τ', there is a unique matrix $S \in \mathcal{S}$ with $Null(S) = Col(N)$ (i.e. the null space of S is the column space of N), and these are all matrices of type τ in $\mathcal{S}$, by the above remark on canonical bases in subspaces of $\mathbb{K}^n$. Hence, there is a matrix $C = C_\tau(N)$ which has 0 entries everywhere except potentially at positions above the pivot positions from $S(N)$, such that $S = S(N) + C_\tau(N) - C_\tau(N)N$. Each such C_τ is a function of N of type τ'. We note that in the case the functions C_τ are all 0, we obtain the semigroup $\mathcal{S}$ of row reduced $n \times n$ matrices over $\mathbb{K}$.

As corollary, for small values of n we can re-obtain a full classification of such Grassmannian semigroups as in [1].

Corollary 2.16. *If $\mathcal{S}$ is a Grassmannian semigroup in $M_2(\mathbb{K})$, then $\mathcal{S}$ is conjugate to $\mathcal{R}$, the semigroup of row reduced matrices.*

Proof. Up to conjugation we may assume $\mathcal{S}$ is in echelon form, and with pivots equal to 1, and therefore, the semigroup is

$$\mathcal{S} = \{I_2; \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, x \in \mathbb{K}; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; 0_2\}$$

where c is an element in $\mathbb{K}$, which is precisely $\mathcal{R}$ (i.e. there is only one Grassmannian semigroup in echelon form and with pivots equal to 1). $\square$

Corollary 2.17. *If $\mathcal{S}$ is a Grassmannian semigroup in $M_3(\mathbb{K})$, then there is a function $f : \mathbb{K} \to \mathbb{K}$ such that $\mathcal{S}$ is conjugate to the following semigroup*

$$\mathcal{S} = \{ \ I_3; \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y \in \mathbb{K}; \begin{pmatrix} 1 & u & f(u) \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, u \in \mathbb{K}; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\begin{pmatrix} 1 & z & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, z, t \in \mathbb{K}; \begin{pmatrix} 0 & 1 & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, w \in \mathbb{K}; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; 0_3. \ \}$$

Proof. We may use the above remark on the structure theorem of such Grassmannian semigroups and note that for each type there are precisely the above families, with the fourth one of type $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, or proceed as follows. Note that each of the following types families of subspaces of V, which exhaust the subspaces of V, must be the kernel of a corresponding matrix (the vectors should be considered as column vectors in $V = \mathbb{K}^3$):

$$\{ \ 0; Span(-x, -y, 1), x, y \in \mathbb{K}; (1, -u, 0); Span(1, 0, 0);$$

$$Span\{(-z, z, 0), (-t, 0, t)\}, z, t \in \mathbb{K}; Span(-w, 1, 0), w \in \mathbb{K}; Span(0, 0, 1); V \ \}$$

We get a semigroup in echelon form with pivots equal to 1, except the second of the above types of matrices, except is in general in the form $\begin{pmatrix} 1 & \alpha & x \\ 0 & 1 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y \in \mathbb{K};$; nevertheless, we know the

matrix E_2 should be in $\mathcal{S}$, and multiplying the two together we get that $\begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}$, and

by the uniqueness property of the Grassmannian semigroup we obtain $\alpha = 0$. Finally, the fourth type is $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and has the property $Ne_3 = e_2 + ce_1$, $Ne_2 = e_1$, $Ne_1 = 0$. After

further changing the basis to $e_3' = e_3$, $e_2' = e_2 + ce_1$, $e_1' = (1 + c)e_1$, we see that the semigroup becomes of the desired form in the basis $\{e_1', e_2', e_3'\}$ (see also Proposition 3.9).

It is easy to see that the above described set is closed under multiplication and forms a Grassmannian semigroup. $\square$

3. THE STRUCTURE OF GRASSMANNIAN SEMIGROUPS

We note a few basic facts on the set of "types" of matrices. We note that the set Π_n of all shapes or "types" has a monoid structure. For each shape $\tau = (k_1, \dots, k_s)$, let P_τ be the matrix having 1 on positions (i, k_i) and 0 elsewhere. It is not difficult to see that the set of $n \times n$ matrices

$\Pi_n = \{P_\tau | \tau \text{ is a shape}\}$ is closed under product. This can be used to introduce a multiplication of "shapes": $\tau\sigma$ is such that $P_{\tau\sigma} = P_\tau P_\sigma$.

We notice also that the type or shape of an element in a Grassmannian semigroup can also be defined without reference to a basis for which it is in echelon form. For this, note that given the flag $0 \subset V_1 \subset \cdots \subset V_n = V$ of images of elements in $\mathcal{S}$, we have $AV_i = V_j$ for some $j \leq i$, and $k_i = \min\{j \mid A(V_j) = V_i\}$ when $V_i = A(V_j)$ for at least one j, equivalently, $V_j \subset \mathrm{Im}(A)$. This is easy to see for $\mathcal{S}$ in any base in which it is in echelon form, so it is independent of such a basis. We will show that the types can be defined independent without reference to the space V on which $\mathcal{S}$ acts.

Remark 3.1. *We also note here a short conceptual proof of the fact that the row reduced matrices are closed under products. Indeed, one can interpret the set of row reduced matrices $\mathcal{R}$ as the set of endomorphisms T of a finite dimensional vector space V, which have the following properties with respect to a fixed basis $\{e_1, \ldots, e_n\}$:*
(1) If $I_k = \mathrm{Span}\{e_1, \ldots, e_k\}$, then for all $k \leq n$, we have $TI_k = I_s$ for some $s \leq k$.
(2) If k is such that $T(I_{k-1}) \subsetneq T(I_k)$, then $T(e_k) \in \{e_1, e_2, \ldots, e_n\}$ (more precisely, $T(e_k) = e_t$ such that $T(I_k) = I_t$). This condition can be written equivalently as follows: if $V_s = \mathrm{Im}(T)$ and $(k_1, \ldots, k_s)$ is such that $k_i = \min\{j \mid T(V_j) = V_i\}$, then $T(e_{k_i}) = e_i$.
The above two conditions make it easy to check that if two endomorphisms A, B satisfy these conditions, then AB satisfies the same conditions as well.
Also, by the results of the first section, we note we have proved that for any Grassmannian semigroup $\mathcal{S}$ there is a flag $I_1 \subset \cdots \subset I_n = \mathbb{K}^n$ on $\mathbb{K}^n$ with respect to which elements of $\mathcal{S}$ have the first property (1). The only difference to row reduced matrices is that, in general, in a Grassmannian semigroup one does not have to have property (2) hold in general.

We prove a few simple propositions on decomposition of elements in a Grassmannian semigroup. As before, $\mathcal{S}$ will denote a Grassmannian semigroup in $M_n(\mathbb{K})$. First we note the following fact regarding solutions of equations in such semigroups.

Proposition 3.2. *Let $\mathcal{S}$ a Grassmannian semigroup and $a, b \in \mathcal{S}$. Then the equation $a = xb$ has a solution in $\mathcal{S}$ if and only if $\ker(b) \subseteq \ker(a)$. Moreover, in this case, there is a unique solution x of maximal rank i.e. with $\mathrm{rank}(x) = n - \mathrm{rank}(b) + \mathrm{rank}(a)$, and $\ker(x) = b(\ker(a))$.*

Proof. Of course, $\ker(b) \subseteq \ker(a)$ is necessary. To show it is sufficient, note that if a solution of $a = xb$ exists then $\ker(a) = b^{-1}(\ker(x))$ (and $b(\ker(a)) \subseteq \ker(x)$). Thus, if $\ker(b) \subseteq \ker(a)$, let $W = b(\ker(a))$ and let $x \in \mathcal{S}$ be such that $\ker(x) = W$. Then $\ker(xb) = b^{-1}(\ker(x)) = b^{-1}(W) = \ker(a)$ (since $\ker(b) \subseteq \ker(a)$), and since $xb, a \in \mathcal{S}$, by uniqueness of kernels we get $xb = a$. By Sylvester's inequality we have $\mathrm{rank}(a) \geq \mathrm{rank}(x) + \mathrm{rank}(b) - n$, so $\mathrm{rank}(x) \leq n - \mathrm{rank}(b) + \mathrm{rank}(a)$. If equality is assumed, then it follows that $\dim(\ker(x)) = \dim(\ker(a)) - \dim(\ker(b))$. This shows that the linear map $b : \ker(a) = b^{-1}(\ker(x)) \to \ker(x)$ is surjective (since $\ker(b) \subseteq \ker(a)$, $\dim(\mathrm{Im}(b|_{\ker(a)})) = \dim(\ker(a)) - \dim(\ker(b)) = \dim(\ker(x))$), and so $\ker(x) = b(\ker(a))$. Therefore, x is unique as it is uniquely determined by its kernel. $\qquad\square$

Next we give a result about unique decompositions of elements. Recall that if W is a vector space, a flag on W is a sequence $0 = W_0 \subset W_1 \subset \cdots \subset W_k = W$ with $\dim(W_i) = i$; a partial flag is a sequence $W_s \subset W_{s+1} \subset \cdots \subset W_t$ with $\dim(W_i) = i$. The next proposition shows that elements decompose uniquely along (partial) flags.

Proposition 3.3. *Let $\mathcal{S}$ be a Grassmannian semigroup, and $a \in \mathcal{S}$.*
(i)Suppose $X_1 \subset X_2 \subset \cdots \subset X_t = \ker(a)$ is a sequence of subspaces. Then there is a unique

decomposition $a = a_1 a_2 \dots a_t$ with $a_i \in \mathcal{S}$ and $\ker(a_{i+1} \dots a_t) = X_{t-i}$ and each a_i is of maximal rank, i.e. $\operatorname{rank}(a_i) = n - \dim(X_i) + \dim(X_{i-1})$.
(ii) If $0 = X_0 \subset X_1 \subset \cdots \subset X_t = \ker(a)$ is a flag on $\ker(a)$, then there is a unique decomposition $a = a_t a_{t-1} \dots a_1$ with $a_i \in \mathcal{S}$ and $\ker(a_i a_{i-1} \dots a_1) = X_i$ and $\operatorname{rank}(a_i) = n - 1$.

Proof. (i) We apply the previous proof recursively. First, write $a = a_1 b_1$ uniquely with $\ker(b_1) = X_{t-1}$ and a_1 of maximal rank equal to $n - \dim(X_{k-1}) + \dim(X_k)$. Then repeat the procedure for b_1 and $X_{t-2} \subset \ker(b_1)$ to obtain $b_1 = a_2 b_2$ with $\ker(b_2) = X_{t-2}$ and a_2 has maximal possible rank, etc. To see uniqueness, if $a = a_1 \dots a_t = a'_1 \dots a'_t$ are two such decompositions, then since $\ker(a_2 \dots a_t) = \ker(a'_2 \dots a'_t)$ then $a_2 \dots a_t = a'_2 \dots a'_t$ and using the uniqueness of the solution x of maximal rank of the equation $a = x a_2 \dots a_t$ provided by the previous proposition we get $a_1 = a'_1$, etc.
(ii) Follows from immediately from (i). $\qquad\square$

Proposition 3.4. *If $\mathcal{S}$ is a Grassmannian semigroup on V of dimension n, $A \in \mathcal{S}$ is of rank k, E is an idempotent of $\operatorname{rank}(E) = s \geq k$, then $EA = A$.*

Proof. As in the beginning, we note $\operatorname{Im}(A) = V_k \subset V_s$, and V_s is the set of 1-eigenvectors of E since E is idempotent. Therefore, $EA(v) = A(v)$ for all $v \in V$. $\qquad\square$

3.1. **Nilpotent elements.** We need one more proposition that describes the nilpotent elements in a Grassmannian semigroup. Recall that given such $\mathcal{S}$ we denoted by V_k the subspace of V which is the (common) image of the elements of rank k in $\mathcal{S}$.

Proposition 3.5. *If $a \in \mathcal{S}$, then there is k such that $a^k = a^{k+1} = \dots$ and a^k is an idempotent.*

Proof. The ascending sequence $(\ker(a^k))_k$ of subspaces of V must stabilize

$$\ker(a^k) = \ker(a^{k+1}) = \dots.$$

By the Grassmannian semigroup property, $a^k = a^{k+1} = \dots a^{2k} = \dots$, and so a^k is also an idempotent. $\qquad\square$

Proposition 3.6. *Let $\mathcal{S}$ be a Grassmannian semigroup. Then the following are equivalent for $x \in \mathcal{S}$:*
(i) x is nilpotent;
(ii) $V_1 \subseteq \ker(x)$
(iii) x is a left zero-divisor in $\mathcal{S}$, i.e. there is $y \in \mathcal{S}$ such that $xy = 0$.

Proof. (i)$\Rightarrow$(ii) If x is nilpotent, let k be such that $x^{k-1} \neq 0 = x^k$. Then $0 \neq \operatorname{Im}(x^{k-1}) \subseteq \ker(v)$. Obviously, $\operatorname{Im}(x^{k-1}) = V_i$, $i \geq 1$, so $V_1 \subseteq \operatorname{Im}(x^{k-1}) \subseteq \ker(x)$.
(ii)$\Rightarrow$(i) Let k be such that $x^k = x^{k+1} = \dots$, so x^k is idempotent. We claim $x^k = 0$. Indeed, otherwise we have $V_1 \subseteq \operatorname{Im}(x^k) = V_i$ for some $i \geq 1$, and since $V_1 \subseteq \ker(x)$, it follows that $\dim(x(V_i)) \leq \dim(V_i) - \dim(V_1) = i - 1 < i = \dim(V_i)$. So $\operatorname{Im}(x^{k+1}) = x(V_i) \neq V_i = \operatorname{Im}(x^k)$, which contradicts $x^k = x^{k+1}$.
(ii)$\Rightarrow$(iii) Obviously, $V_1 \subseteq \ker(x)$ if and only if $V_1 = \operatorname{Im}(y) \subseteq \ker(x)$ for every $y \in \mathcal{S}$ of rank 1 (and there are such elements in $\mathcal{S}$), which is equivalent to $xy = 0$ for all such elements y. $\qquad\square$

In the proposition above we may easily see that right zero divisors are not necessarily nilpotent: if $b \in \mathcal{S}$ is such that $\operatorname{Im}(b) = V_{n-1}$ and $\ker(b) = Y \neq V_1$, and $a \in \mathcal{S}$ is such that $\ker(a) = V_{n-1}$, then $ab = 0$ so b is a right zero-divisor, but b is not nilpotent since $V_1 \not\subset \ker(b)$.
By extension of the terminology of rings, we may call a subset I of a semigroup M an ideal if for all $a \in M$ and $x \in I$, we have $ax, xa \in I$. Of course, this is not going to produce a congruence

relation on M that would be suitable for doing a quotient, as it is for rings. For a Grassmannian semigroup $\mathcal{S}$ we denote by $N(\mathcal{S})$ the set of nilpotent elements of $\mathcal{S}$. This makes sense in any semigroup where there is a "zero" element (i.e. an element z such that $az = za = z$ for all a). We note the following

Proposition 3.7. *Let $\mathcal{S}$ be a Grassmannian semigroup. Then the set of nilpotent elements $N(\mathcal{S})$ is a "prime" ideal of $\mathcal{S}$, namely, if $ab \in N(\mathcal{S})$ then $a \in \mathcal{S}$ or $b \in \mathcal{S}$.*

Proof. This property is easiest visualized in matrix form. Consider a basis with respect to which the semigroup $\mathcal{S}$ is in echelon form (with pivots equaling 1). In particular, $\mathcal{S} \subset T_n$, the algebra of upper triangular matrices, and if N_n is the set of strictly upper triangular matrices, then $N(\mathcal{S}) = N_n \cap \mathcal{S}$. Since N_n is an ideal of T_n (the Jacobson radical of T_n), it follows that $N(\mathcal{S})$ is an ideal of $\mathcal{S}$.

For "primality", let $c \notin N(\mathcal{S})$, and $a \in \mathcal{S}$. Then c must have its entry on position $(1,1)$ equal to 1. This can be seen either from Proposition 3.6, or from the obvious fact that matrices in echelon form are nilpotent if and only if they have a 0 on position (1,1). Now the following equalities can happen only if $a_1 = a_2 = 0$, and this shows that if either ac or ca is nilpotent, then so is a.

$$
\begin{pmatrix} 1 & * & \ldots & * \\ 0 & * & \ldots & * \\ & \ldots & \ldots & \\ 0 & \ldots & 0 & * \end{pmatrix} \cdot \begin{pmatrix} a_1 & * & \ldots & * \\ 0 & * & \ldots & * \\ & \ldots & \ldots & \\ 0 & \ldots & 0 & * \end{pmatrix} = \begin{pmatrix} 0 & * & \ldots & * \\ 0 & 0 & \ldots & * \\ & \ldots & \ldots & \\ 0 & \ldots & 0 & 0 \end{pmatrix} =
$$

$$
= \begin{pmatrix} a_2 & * & \ldots & * \\ 0 & * & \ldots & * \\ & \ldots & \ldots & \\ 0 & \ldots & 0 & * \end{pmatrix} \cdot \begin{pmatrix} 1 & * & \ldots & * \\ 0 & * & \ldots & * \\ & \ldots & \ldots & \\ 0 & \ldots & 0 & * \end{pmatrix}
$$

$\square$

Remark 3.8. *Let $N_1 \in \mathcal{S}$ be such that $\ker(N_1) = V_1$. Since $V_1 \subseteq \ker(x)$ for all $x \in N(\mathcal{S})$, we see that for each such $x \in N(\mathcal{S})$ there is $a \in \mathcal{S}$ such that $x = aN_1$, i.e. N_1 divides all the nilpotent elements of $\mathcal{S}$. This particular nilpotent will be important next.*

We denote N_k the nilpotent element of $\mathcal{S}$ for which $\ker(N_k) = I_k$. Let $e_n \in V_n \setminus V_{n-1}$. Since $\text{rank}(N_1) = n - 1$, we have that $N_1^n = 0 \neq N_1^{n-1}$. Define the elements e_i by $e_{n-1} = N_1(e_n)$, ... $e_{i-1} = N_1(e_i)$, $i \geq 1$, so $e_0 = 0$. We claim that $(e_i)_{i=1,\ldots,n}$ is a basis of V, and, moreover, $\{e_1, \ldots, e_k\}$ is a basis of V_k for each k. For this it is enough to show that $e_i \in V_i \setminus V_{i-1}$ for all i. Suppose $e_k \in V_{k-1}$ for some k, and let k be the largest such number; obviously, $k < n$ since $e_n \notin V_{n-1}$. We have that $N_1(V_i) = V_{i-1}$ for all i. Then $e_{k+1} \notin V_k$, so $V_{k+1} = \mathbb{K}e_{k+1} + V_k$, and therefore $N_1(V_{k+1}) = \mathbb{K}N_1(e_{k+1}) + N_1(V_k) = \mathbb{K}e_k + N_1(V_k) \subseteq V_{k-1}$. But this is a contradiction to $N_1(V_{k+1}) = V_k$, so the claim is proved. With this we have the following

Proposition 3.9. *With the above notations, $N_1^k = N_k$. Moreover, there is a basis of V with respect to which, in the semigroup $\mathcal{S}$, N_k is the matrix with 1 on the k'th diagonal above the main diagonal, and 0 elsewhere, so N_1 is the Jordan cell of dimension n and eigenvalue 0.*

Proof. Consider the basis above $\{e_1, \ldots, e_n\}$. Obviously, in this basis N_1 is a Jordan cell of dimension n and eigenvalue 0. Moreover, $\ker(N_1^k) = Span\{e_1, \ldots, e_k\} = V_k$, since $\dim(\ker(N_1^k)) = k$ and it is straightforward to note that $N_1^k(e_i) = 0$ when $i \leq k$. Hence, by the uniqueness of the elements with a given kernel in a Grassmannian semigroup, $N_1^k = N_k$. $\square$

A few remarks on intrinsic abstract properties of Grassmannian semigroups. We note that several invariants of a Grassmannian semigroup $\mathcal{S}$ can be defined without any reference to the action of $\mathcal{S}$ on V.

(I1) First, note that the identity I and zero 0 elements of $\mathcal{S}$ are uniquely determined.

(I2) Next, the element N_1 is uniquely determined by the property that for all nilpotent $x \in S$, there is $y \in \mathcal{S}$ such that $x = yN_1$ (since such an element in $\mathcal{S}$, considered as an endomorphism of V, will have its kernel contained in the kernel of all nilpotent elements in $\mathcal{S}$).

(I3) The structure of idempotents is uniquely determined. Let $\sim$ be the equivalence relation on the set $E(\mathcal{S})$ of idempotents of $\mathcal{S}$ given by $E \sim E'$ if $EE' = E'$ and $E'E = E$. If $\mathcal{S} \subset \operatorname{End}_{\mathbb{K}}(V)$ is any fixed representation of $\mathcal{S}$ as Grassmannian semigroup on V, it is easy to see that $E \sim E'$ if and only if $\operatorname{rank}(E) = \operatorname{rank}(E')$. Thus, n - the dimension of the vector space V equals the number of equivalence classes of idempotents. We may also introduce a quasi-ordering on $E(V)$ by setting $E' \geq E$ if and only if $EE' = E'$, which is equivalent to $\operatorname{Im}(E') \subseteq \operatorname{Im}(E)$, and further $\operatorname{rank}(E') \leq \operatorname{rank}(E)$. This becomes a total order on $E(\mathcal{S})/\sim$, making it a PO-set isomorphic to $\{0, 1, \ldots, n\}$. Let H_k be the equivalence class of level k (corresponding to idempotents of rank k).

(I4) The structure of idempotents determines a filtration on $\mathcal{S}$ that recovers rank. Namely, using Proposition 3.4, we see that if $A \in \mathcal{S}$, then $\operatorname{rank}(A) \leq k$ if and only if $EA = A$ for some $E \in H_k$, equivalently, for all $E \in E_k$. Hence, we may introduce the subset $R_k = R_k(\mathcal{S})$ consisting of elements A for which $EA = A$ for some (equivalently, all) $E \in H_k$. This will consist of all elements of rank $\leq k$. Now, the rank can be defined abstractly as $\operatorname{rank}(A) = k$ if $A \in R_k \setminus R_{k-1}$. This shows that n - the dimension of the space on which $\mathcal{S}$ acts, is recovered as the cardinality of the set of equivalence classes of idempotents $E(\mathcal{S})/\sim$.

(I5) Now, the type of an element in $\mathcal{S}$ can also be defined abstractly. Fix $E_1, E_2, \ldots, E_n$ representatives of $H_1, H_2, \ldots, H_n$ respectively. Given $A \in \mathcal{S}$, consider the sequence of numbers $\operatorname{rank}(AE_1), \ldots, \operatorname{rank}(AE_n)$; in matrix interpretation, the images of these elements correspond to $A(V_1), \ldots, A(V_n)$. Then one defines $k_i = \min\{j \mid \operatorname{rank}(AV_j) = i\}$. It is easy to see that this is an equivalent reformulation of the type of A given before in the case A is in echelon form.

(I6) Note that the type of an element in $\mathcal{S}$ can also be recovered by using the distinguished nilpotent N_1. It is based on the following easy observation: if A is a matrix and N_1 is the Jordan cell with eigenvalue 0 of dimension n, then AN_1 is obtained by deleting the last column of A, shifting the other columns of A to the right and replacing the first one by 0. Hence, the last columns in the matrices in the sequence $A, AN_1, AN_1^2, \ldots, AN_1^{n-1}$ are all the columns of A. If A is an echelon matrix of type $(k_1, \ldots, k_t)$, then it is easy to see that the type of AN_1 is $(k_1 + 1, k_2 + 1, \ldots, k_t + 1)$ if $k_t < n$ and $(k_1 + 1, \ldots, k_{t-1} + t)$ if $k_t = n$. Therefore, the non-increasing sequence $rank(A), rank(AN_1), \ldots, rank(AN_1^{n-1})$ will completely determine the type of the echelon matrix A: the ranks will decrease precisely at positions $n - k_t + 1, n - k_{t-1} + 1, \ldots, n - k_1 + 1$. Hence, since rank is intrinsically determined in a Grassmannian semigroup $\mathcal{S}$, this is another way to get the type of an element in $\mathcal{S}$ without reference to the ambient space.

While we defined the Grassmannian semigroups as systems of representatives for the left $Gl_n(\mathbb{K})$ action on $M_n(\mathbb{K})$, it is natural to ask what is their relationship with the right action. This by the next proposition which uses the above results on the structure of such semigroups, and shows that under the right action Grassmannian semigroups are contained in a small (finite) number of orbits.

Proposition 3.10. *Let $\mathcal{S}$ be a Grassmannian semigroup, and $E_k \in \mathcal{S}$ the previously defined basic idempotents. Then $\mathcal{S} \subset \bigcup_{i=0}^{n} E_k Gl_n(\mathbb{K})$; in particular, up to equivalence under right action, there are exactly $n+1$ classes of elements in $\mathcal{S}$.*

Proof. This is obvious, since the right $Gl_n(\mathbb{K})$ actions operates on columns, so preserves column space, and equivalence classes are determined precisely by column space. We have already shown there are exactly $n+1$ possible column spaces for elements in $\mathcal{S}$. $\qquad\square$

4. Isomorphisms of Grassmannian semigroups

In what follows, we aim to study when two Grassmannian semigroups are isomorphic. By the remarks of the previous section, we note that if two such semigroups are isomorphic, then they have the same "dimension", i.e. they are Grassmannian semigroups on the same vector space of dimension n (since n is determined by the internal structure of the semigroup as we saw in the previous section). The first step is to notice that an isomorphism of such semigroups produces an order preserving isomorphism of the lattice of subspaces of the vector space. We will denote $\mathcal{L}(X)$ for the lattice of subspaces of the vector space X. Also, if $\mathcal{S}$ is a Grassmannian semigroup on the vector space V, for each $X \in \mathcal{L}(V)$ denote $a_X \in \mathcal{S}$ the element for which $\ker(a_X) = X$.

Proposition 4.1. *Let $\mathcal{S}, \mathcal{S}'$ be Grassmannian semigroups in $M_n(\mathbb{K})$, and let $\varphi : \mathcal{S} \to \mathcal{S}'$ be an isomorphism. Let $\mathcal{L}(\mathbb{K}^n)$ be the set of subspaces of $\mathbb{K}^n$ and let $p : \mathcal{L}(\mathbb{K}^n) \to \mathcal{L}(\mathbb{K}^n)$ be defined by $p(X) = W$ if and only if $\varphi(a_X) = a'_W$, where $a_X \in \mathcal{S}$ and $a'_W \in \mathcal{S}'$ are the unique elements with $\ker(a_X) = X$, $\ker(a'_W) = W$. Then p is an inclusion preserving bijection.*

Proof. If $X \subseteq Y \subseteq \mathbb{K}^n$ are subspaces, then there is $b \in \mathcal{S}$ such that $a_Y = b a_X$, so $\varphi(a_Y) = \varphi(b)\varphi(a_X)$ by Proposition 3.2; thus $\ker(\varphi(a_X)) \subseteq \ker(\varphi(a_Y))$, and so $p(X) \subseteq p(Y)$. $\qquad\square$

Since the above induced map p is an inclusion preserving bijection on $\mathcal{L}(V)$, we are in position to use the Fundamental Theorem of Projective Geometry and the Skolem-Noether theorem to characterize this map and obtain insights on the isomorphism class of a semigroup. We fix a basis $V = \mathbb{K}^n$ and identify $\mathrm{End}_{\mathbb{K}}(V) = M_n(\mathbb{K})$. For an automorphism σ of the field $\mathbb{K}$, denote $\overline{\sigma} : V \to V$ by applying σ component wise. By extension (and abuse of notation), we will also denote $\overline{\sigma} : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ the ring automorphism obtained by applying σ to each entry of a matrix. Let also $\theta \in Gl_n(\mathbb{K})$. Then the composition map $\tau : A \to \theta A \theta^{-1} \to \overline{\sigma}(\theta A \theta^{-1})$ is a semi-linear automorphism of the ring $M_n(A)$, and it is well known that every semi-linear transformation is obtained this way (recall that an auto-morphism α of the ring $M_n(\mathbb{K})$ is said to be semilinear if $\alpha(c \cdot A) = \sigma(c)\alpha(A)$ for some automorphism σ of the field $\mathbb{K}$). Then $\tau(\mathcal{S})$ is also a Grassmannian semigroup, and we introduce the following

Definition 4.2. *We say that two Grassmannian semigroups $\mathcal{S}, \mathcal{S}'$ are semi-conjugate if $\mathcal{S}' = \tau(\mathcal{S})$ for a semi-linear transformation τ as above.*

In what follows, we will show that two Grassmannian semigroups are isomorphic then they are "almost" semi-conjugate, except for some trivial way of obtaining new Grassmannian semigroups by multiplying matrices by certain constants. We first observe

Proposition 4.3. *Let $\mathcal{S}, \mathcal{S}'$ be two isomorphic Grassmannian semigroups. Then there exists a Grassmannian semigroup $\mathcal{S}_0$ which is semi-conjugate to $\mathcal{S}$ and an isomorphism $\psi : \mathcal{S}_0 \to \mathcal{S}'$ which is kernel preserving, that is, $\ker(\psi(x)) = \ker(x)$ for all $x \in \mathcal{S}_0$.*

Proof. Let $p : \mathcal{L}(\mathbb{K}^n) \to \mathcal{L}(\mathbb{K}^n)$ be the map from Proposition 4.1 induced by the isomorphism $\varphi : \mathcal{S} \to \mathcal{S}'$, which is inclusion preserving. By the Fundamental Theorem of Projective Geometry, we have that p is given by a semi-linear automorphism of $M_n(\mathbb{K})$, so $p(W) = \overline{\sigma}(\theta(W))$, for all subspaces W of $\mathbb{K}^n$, where we interpret by θ as an endomorphism of $\mathbb{K}^n$ via the fixed identification $M_n(\mathbb{K}) = \mathrm{End}_{\mathbb{K}}(V)$. Let $\tau(A) = \overline{\sigma}(\theta A \theta^{-1})$. Then $\mathcal{S}_0 \tau(\mathcal{S})$ is a Grassmannian semigroup which is semi-conjugate to $\mathcal{S}$. Note now that for $a_W \in \mathcal{S}$ (so $\ker(a_W) = W$), we have $\ker(\tau(a_W)) = p(W)$. Indeed, let $v \in \ker(\tau(a_W))$; this is equivalent to $\overline{\sigma}(\theta a_W \theta^{-a})v = 0$ and further to $(\overline{\sigma})^{-1}(v) \in \ker(\theta a_W \theta^{-1})$, i.e. $\theta^{-1}(\overline{\sigma})^{-1}(v) \in \ker(a_W) = W$. Hence, $v \in \ker(\tau(a_W))$ if and only if $v \in \overline{\sigma}(\theta(W))$.

Lastly, if $\psi = \varphi \circ \tau^{-1}$, then the map induced by ψ on $\mathcal{L}(\mathbb{K}^n)$ takes $p(W)$ to $p(W)$ for each $W \in \mathcal{L}(\mathbb{K}^n)$, and so it is kernel preserving. $\square$

Next, we determine what kernel preserving isomorphisms between Grassmannian semigroups look like. We will need the following small Lemma, which may be known, but we could not find a reference.

Lemma 4.4. *Let b, c be linear transformations of V to V' vector spaces of finite dimension such that the maps b^{-1}, c^{-1} on $\mathcal{L}(V')$ are equal. Then there is $\lambda \in \mathbb{K}$, $\lambda \neq 0$ such that $b = \lambda c$.*

Proof. First, $\ker(b) = b^{-1}(0) = c^{-1}(0) = \ker(c)$, so it is easy to see that by factoring out by $\ker(b) = \ker(c)$, we may assume that b and c are injective, since if the induced maps $B = \overline{b}, C = \overline{c}$ will have $B = \lambda C$, it follows immediately that $b = \lambda c$. Also, note that $0 \neq w \in \mathrm{Im}(b)$ if and only if $b^{-1}(\mathbb{K}w) \neq 0$. Since $b^{-1}(\mathbb{K}w) = c^{-1}(\mathbb{K}w)$ for all w, this shows that $\mathrm{Im}(b) = \mathrm{Im}(c)$. Thus, we may also assume $b \neq 0 \neq c$. Let $w_1, \dots, w_n$ be a basis on $\mathrm{Im}(b) = \mathrm{Im}(c)$, and x_i, y_i be such that $b(x_i) = w_i = c(y_i)$. By injectivity, we have $b^{-1}(\mathbb{K}w_i) = \mathbb{K}x_i$ and $c^{-1}(\mathbb{K}w_i) = \mathbb{K}y_i$, and the hypothesis thus implies $\mathbb{K}x_i = \mathbb{K}y_i$ so $y_i = \lambda_i x_i$. Let $W = \mathbb{K}(w_i + w_j)$ for $i \neq j$. Since $b(x_i + x_j) = w_i + w_j = c(y_i + y_j)$, by the injectivity of b and c and hypothesis we get $\mathbb{K}(x_i + x_j) = b^{-1}(W) = c^{-1}(W) = \mathbb{K}(y_i + y_j)$, so $y_i + y_j = \lambda(x_i + x_j)$ for some λ. Hence, $\lambda_i x_i + \lambda_j x_j = \lambda x_i + \lambda x_j$, and since x_i, x_j are linearly independent (since b is injective and w_i, w_j are independent), we get $\lambda_i = \lambda = \lambda_j$. This shows that $\lambda_1 = \cdots = \lambda_n$, so $y_i = \lambda x_i$ and therefore $b(x_i) = w_i = c(y_i) = c(\lambda x_i) = \lambda c(x_i)$, which shows that $b = c$. $\square$

Proposition 4.5. *Let $\varphi : \mathcal{S} \to \mathcal{S}'$ be a kernel preserving isomorphism between two Grassmannian semigroups on the n-dimensional vector space V, $n \geq 2$. Let $N(\mathcal{S})$ be the set of nilpotent elements in $\mathcal{S}$ as before. Then:*
(i) If $n = 2$, then there is $\lambda \in \mathbb{K}$ such that $\mathcal{S}' = \mathcal{S} - \{N_1\} \cup \{\lambda N_1\}$, and

$$\varphi(a) = \begin{cases} a & \text{if } a \notin N(\mathcal{S}), \\ \lambda \cdot a & \text{if } a = N_1 \end{cases}$$

(ii) $\mathcal{S} = \mathcal{S}'$ and $\varphi = \mathrm{Id}$ if $n \geq 3$.

Proof. Since φ is kernel preserving, we have $\ker(b) = \ker(\varphi(b))$ for all $b \in \mathcal{S}$. Let W be a subspace of V, let $b \in \mathcal{S}$, and let $a = a_W \in \mathcal{S}$ so that $\ker(a) = W$. Notice that

$$
\begin{aligned}
b^{-1}(W) &= b^{-1}(\ker(a)) \\
&= \ker(ab) = \ker(\varphi(ab)) \quad \text{(since φ is kernel preserving)} \\
&= \ker(\varphi(a)\varphi(b)) \quad \text{(since φ is a morphism)} \\
&= \varphi(b)^{-1}(\ker(\varphi(a))) \\
&= \varphi(b)^{-1}(\ker(a)) \quad \text{(since φ is kernel preserving)} \\
&= \varphi(b)^{-1}(W)
\end{aligned}
$$

Therefore, b^{-1} and $\varphi(b)^{-1}$ are equal on $\mathcal{L}(V)$ and by the previous Lemma $\varphi(b) = \lambda(b) \cdot b$ for some $\lambda(b) \in \mathbb{K} \setminus \{0\}$.

Note that if $b, c \in \mathcal{S}$ are such that $bc \neq 0$, then $\lambda(bc) = \lambda(b)\lambda(c)$: indeed, $\varphi(bc) = \lambda(bc)bc = \varphi(b)\varphi(c) = \lambda(b)\lambda(c)bc$ and $bc \neq 0$. Next, if $e \in \mathcal{S}$ is a nonzero idempotent, then $\lambda(e)^2 = \lambda(e)$ in $\mathbb{K}$ and $\lambda(e) \neq 0$ so $\lambda(e) = 1$. Furthermore, if $a \notin N(\mathcal{S})$, then there is k such that $a^k = a^{k+1} \neq 0$ and a^k is an idempotent (Proposition 3.5). Hence $\lambda(a^k) = 1$ and $\lambda(a^k) = \lambda(a \cdot a^k) = \lambda(a)\lambda(a^k)$ so $\lambda(a) = 1$.

Let x be a nilpotent element, so $V_1 \subseteq \ker(x)$ by Proposition 3.6. If x is nilpotent with $x \neq N$, then $V_1 \subsetneq \ker(x)$, and let Y be a subspace of $\ker(x)$ of co-dimension 1 and such that $V_1 \not\subset Y$. Let $a = a_Y$ (so $\ker(a) = Y$) and let $c \in \mathcal{S}$ be such that $x = ca$ (it exists since $\ker(a) \subset \ker(x)$). Moreover, we may assume c has maximal rank equal to $n-1$ since $\dim(\ker(x)) - \dim(\ker(a)) = 1$, and there is a unique such c by Proposition 3.2. Then a is not nilpotent since $V_1 \not\subset Y$, and so $c \in N(\mathcal{S})$ by Proposition 3.7. Thus, $V_1 \subseteq \ker(c)$, and so $V_1 = \ker(c)$ (since $\text{rank}(c) = n-1$), and therefore $c = N_1$. Thus $0 \neq x = N_1 a$, so $\lambda(x) = \lambda(N_1)\lambda(a)$, and as $\lambda(a) = 1$ (since $a \notin N(\mathcal{S})$), we get $\lambda(x) = \lambda(N_1)$, and so λ is constant on $N(\mathcal{S})$.

Finally, if $n = 2$ there is only one non-zero nilpotent element, and the statement (i) follows. Otherwise, we have $N_1^2 \neq 0$, since $\text{rank}(N_2) = n-1$ so $\text{rank}(N_1^2) \geq n-1+n-1-n = n-2 \geq 1$ ($n = \dim(V) \geq 3$). Hence, $\lambda = \lambda(N_1^2) = \lambda(N_1)\lambda(N_1) = \lambda^2$, and as $\lambda \neq 0$, we get $\lambda = 1$. The conclusion of (ii) follows. $\qquad\square$

Since for $n = 2$ every two Grassmannian semigroups are conjugate by Corollary 2.16, we have the following:

Corollary 4.6. *Two Grassmannian semigroups are isomorphic if and only if they are semi-conjugate.*

In particular we have:

Corollary 4.7. *If $\mathbb{K}$ is such that $\text{Aut}(\mathbb{K}) = \{\text{Id}_{\mathbb{K}}\}$, then two Grassmannian semigroups are isomorphic if and only if they are conjugate. In particular, two real Grassmannian semigroups are isomorphic if and only if they are conjugate (since $\text{Aut}(\mathbb{R}) = \{\text{Id}\}$).*

4.1. **Small dimensions.** By the previous section, every two Grassmannian semigroups on a vector space of dimension 2 are isomorphic. We aim to study this problem in dimension 3. Since every Grassmannian semigroup is conjugate to one in echelon form, we will investigate when two Grassmannian semigroups in echelon form are isomorphic, and when they are conjugate. We note that if $\mathcal{S}, \mathcal{S}'$ are Grassmannian semigroups in echelon form, and $\varphi : \mathcal{S} \to \mathcal{S}'$ is an isomorphism such that $\varphi(X) = \theta\overline{\sigma}(X)\theta^{-1}$ for invertible θ, then $\mathcal{S}_0 = \overline{\sigma}(\mathcal{S})$ is also a semigroup in echelon form, and so $\mathcal{S}_0$ and $\mathcal{S}'$ are conjugate.

Proposition 4.8. *Let $\mathcal{S}_0$ and $\mathcal{S}'$ be Grassmannian semigroups in echelon form that are conjugate by θ. Then $\theta = p(J_n(0))$, where p is a polynomial and $J_n(0) = N_1$ is the Jordan cell of dimension n and eigenvalue 0.*

Proof. Since the element N_1 is in both $\mathcal{S}_0$ and $\mathcal{S}'$, and it is uniquely defined by the internal semigroup structure and invariant properties described above - invariant property (I2), then the conjugation isomorphism $X \mapsto \theta X \theta^{-1}$ must take N_1 to N_1. Hence, $\theta N_1 = N_1 \theta$. But it is well known (and computationally straightforward to check) that the centralizer of the Jordan cell $J_n(0)$ consists of polynomial functions of $J_n(0)$, which ends the proof. $\square$

We apply the previous proposition to determine Grassmannian semigroups of dimension 3 up to isomorphism. As noted in [1] and Corollary 2.17 above, they are completely determined by a function f. We determine first when two such semigroups are conjugate. If $\mathcal{S}(f)$, $\mathcal{S}(g)$ are two such semigroups associated with the functions $f, g : \mathbb{K} \to \mathbb{K}$, and conjugation by θ is an isomorphism between them, then $\theta = \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$ as noted in the previous proposition (we may assume the diagonal is 1, since we may always multiply the conjugation matrix by a scalar, since conjugation by diagonal matrices has no effect). Conjugation will preserve the type (since type is an intrinsic property of the semigroup structure), and we have

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & w & f(w) \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} 1 & w & f(w)+a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -a & a^2-b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -a+w & a^2-b-aw+f(w)+a \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$$

and therefore we obtain $g(w-a) = a^2 - b - aw + f(w) + a$, or, equivalently, $g(t) = f(t+a) - at + a - b$ for all $t \in \mathbb{K}$. Furthermore, it is easy to see that two Grassmannian semigroups $\mathcal{S}(f), \mathcal{S}(g)$ in echelon form are isomorphic via an isomorphisms of type $\bar{\sigma}$ for $\sigma \in \mathrm{Aut}\,(\mathbb{K})$ if $\sigma(f(w)) = g(\sigma(w))$ for all $w \in \mathbb{K}$. Thus, combining the two, we get the following, that recovers in particular another result of [1].

Proposition 4.9. *Let $\mathcal{S}(f)$, $\mathcal{S}(g)$ be two Grassmannian semigroups in echelon form as in Corollary 2.17, with $f, g : \mathbb{K} \to \mathbb{K}$. Then:*
(i) $\mathcal{S}(f), \mathcal{S}(g)$ are conjugate if and only if $g(t) = f(t+a) - at + a - b$, $\forall t \in \mathbb{K}$ for some $a, b \in \mathbb{K}$.
(ii) $\mathcal{S}(f)$ and $\mathcal{S}(g)$ are isomorphic if and only if $g(w) = \sigma(f(\sigma^{-1}(w)) + a) - a\sigma^{-1}(w) + a - b$, $\forall w \in \mathbb{K}$ for some $a, b \in \mathbb{K}$ and $\sigma \in \mathrm{Aut}\,(\mathbb{K})$.

Denote by $U_3(\mathbb{K})$ the group of unipotent upper triangular matrices of the above type $\begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$, and $Fun(\mathbb{K}, \mathbb{K})$ the set of maps from $\mathbb{K}$ to $\mathbb{K}$. Obviously, $U_3(\mathbb{K})$ is isomorphic to the quotient of the group of units of $\mathbb{K}[X]/(X^3)$ by the scalars $\lambda \in \mathbb{K}^\times$, and it is abelian. Also, $\mathrm{Aut}\,(\mathbb{K})$ acts on this abelian group in the obvious way (acting on each entry). Thus their semidirect product $\mathrm{Aut}\,(\mathbb{K}) \rangle U_3(\mathbb{K})$ acts on $Fun(\mathbb{K}, \mathbb{K})$ by the action described in (ii) of the

above proposition, and by the above remarks the orbits of this action parametrize the set of isomorphism types of Grassmannian semigroups in $M_3(\mathbb{K})$. The cardinality of the group $U_3(\mathbb{K})$ is obviously that of $\mathbb{K}$ if $\mathbb{K}$ is infinite, and the cardinality of $Fun(\mathbb{K}, \mathbb{K})$ is $|\mathbb{K}|^{|\mathbb{K}|}$, which is larger than $|\mathbb{K}|$. In particular, when $\mathrm{Aut}(\mathbb{K})$ is not too large, we can easily obtain:

Corollary 4.10. *If $\mathbb{K}$ is an infinite field with $\mathrm{Aut}(\mathbb{K}) = \{Id\}$, or, more generally, if $|\mathrm{Aut}(\mathbb{K})| < |\mathbb{K}|$, then there are $|\mathbb{K}|^{|\mathbb{K}|}$ isomorphism types of Grassmannian semigroups of dimension 3. In particular, when $\mathbb{K} = \mathbb{R}$, the set of isomorphism types of Grassmannian semigroups of dimension 3 has cardinality $\aleph_2 = 2_1^{\aleph} = 2^{2^{\aleph_0}}$.*

Given all the above results on the structure of Grassmannian semigroups, one may certainly ask whether there is an algebraic feature of the semigroup $\mathcal{R}$ of row reduced matrices (either an internal one or one relative to the ambient space $M_n(\mathbb{K})$ and action on $\mathcal{L}(\mathbb{K}^n)$) which distinguishes $\mathcal{R}$ from all other semigroups. Thus, we formulate

Question. Give a characterization of the semigroup $\mathcal{R}$ among all Grassmannian semigroups.

One such characterization is given by Remark 3.1; we mention it below without proof, which can be deduced easily from that 3.1. The following theorem states that the semigroup of row reduced matrices is that for which the type of an element can be read of a particular fixed basis.

Theorem 4.11. *Let $\mathcal{S}$ be a Grassmannian semigroup. Then $\mathcal{S}$ is isomorphic to $\mathcal{R}$ if and only if there is a basis $\{e_1, \ldots, e_n\}$ such that for every element $A \in \mathcal{S}$, if $\tau = (k_1, \ldots, k_s)$ is the type of A, then $A(e_{k_i}) = e_i$.*

Another perhaps not so remarkable characterization that parallels Proposition 3.10 is the following: a semigroup $\mathcal{S}$ which is in echelon form equals the semigroup of row reduced matrices if for every element $A \in \mathcal{S}$, there is a permutation matrix P such that AP is an idempotent.

We note a third characterization of $\mathcal{R}$, somewhat in the same spirit as the previous two. We make the following remark: if B is an echelon matrix of type $\tau = (k_1, \ldots, k_t)$, then for $p \in \{k_1, \ldots, k_t\}$, say $p = k_i$, we have that $BE_p - BE_{p-1}$ is a matrix whose only nonzero column is the p'th column which equals the p'th column of B. In order to have that B is in row reduced form, this column needs to have just one non-zero element equal to 1 at position (i, k_i). Since left multiplication of B by the elements E_j selects the first j lines and replaces the rest by 0, this can be tested by asking that $E_j B = 0$ for $j < i$. Hence, B is in row reduced form if $E_j B E_{k_i} = E_j B E_{k_i-1}$ for all $j < k_i$.

Proposition 4.12. *Let $\mathcal{S}$ be a Grassmannian semigroup. Let $\mathcal{S}'$ be a Grassmannian semigroup in echelon form which is conjugated to $\mathcal{S}$ and let $F_i \in \mathcal{S}$ be elements corresponding to $E_i \in \mathcal{S}'$ via this conjugation. Then $\mathcal{S}$ is conjugated to the semigroup of row reduced matrices if and only if for each $B \in \mathcal{S}$ of type $\tau = (k_1, \ldots, k_t)$, we have $E_j B E_{k_i} = E_j B E_{k_i-1}$ for all $j < k_i$.*

It is natural to ask whether it is possible to give a characterization of the semigroup of row reduced matrices, which is an intrinsic algebraic characterization, independent of the embedding into the ambient matrix algebra. The above proposition could offer some clues on the possibility of such a characterization. In fact, if the "basic" matrix idempotents E_i that appear in any Grassmannian semigroup in echelon form could be characterized intrinsically only in terms of the properties of the semigroup, the above proposition would then offer such a characterization. Unfortunately, this seems to be hard to achieve. For this, note that if $\mathcal{S}$ is a Grassmannian semigroup in echelon form, then two matrices $A, B \in \mathcal{S}$ that have the same first $n - 1$ columns are indistinguishable by right multiplications (except by identity). That is, if $C \in \mathcal{S}$, $C \neq I$,

then the last row of C is 0, so $AC = BC$. Left multiplications on the other hand at a glance seem to be quite general. In fact, for example, as far as elements of rank 1 are concerned, left multiplication does not help either, since for any such element $A \in \mathcal{S}$ with $rank(A) = 1$, we get $CA = 0$ or $CA = A$ for all $C \in \mathcal{S}$.

5. Graded algebra structure on Grassmannians and semigroups

We describe a connection between Grassmannian semigroups and a certain graded algebra structure on Grassmannians. Recall that by $Gr_{\mathbb{K}}(k, n)$ or $Gr(k, n)$ one denotes the set of subspaces of $\mathbb{K}^n$, the set of Grassmannians. Recall that Π_n has a monoid structure. Let $Gr_{\mathbb{K}}(n)$ be the set of all subspaces of the space $\mathbb{K}^n$, that is, the "total" Grassmannian. One can write $Gr_{\mathbb{K}}(n) = \bigcup_{\tau \in \Pi_n} W_\tau$, which can be regarded as a bijection obtained by giving each subspace of $\mathbb{K}^n$ a canonical basis of column (or row) reduced vectors (the basis $e_1, \ldots, e_n$ is fixed). Recall that W_τ is the set of matrices of the type $A - P_\tau$, where A is a row reduced matrix of type τ. Since any subspace of $\mathbb{K}^n$ regarded as line vectors has a unique row reduced basis, we see that $Gr_{\mathbb{K}}(n, t) = \bigcup_{\tau = (k_1, \ldots, k_t)} W_\tau$, and each W_τ corresponds to some Schubert cell. Recall that if $\tau = (k_1, \ldots, k_t)$, then W_τ is a subspace of $M_n(\mathbb{K})$ and $\dim(W_\tau) = t(n - t) + t(t + 1)/2 - (k_1 + \cdots + k_t)$, so we may view each W_τ as an affine space of appropriate dimension. If t is fixed this is maximum when $(k_1, \ldots, k_t) = (1, \ldots, t)$. This agrees with the known fact that the dimension of the Grassmannian $G_{\mathbb{K}}(n, t)$ is $t(n - t)$.

Let $\mathcal{S}$ be a Grassmannian semigroup. By Theorem 2.15 we may assume, after possible conjugation by a matrix, that $\mathcal{S}$ is in echelon form. Using either Theorem 2.15 or directly the definition of Grassmannian semigroup and the remarks of the preceding sections, the set of row reduced matrices of $\mathcal{S}$ of type τ is in 1-1 correspondence with the space $W_{\tau'}$ and with W_τ. Let $\psi_\tau : W_\tau \to \mathcal{S}$ be a bijection which parametrizes these matrices; this may be taken to be algebraic. It is not difficult to observe that if A is an echelon matrix of type τ (with pivots equal to 1) and B is an echelon matrix of type σ (with pivots equal to 1), then AB is an echelon matrix of type $\tau\sigma$ (with pivots equal to 1). Hence, the semigroup $\mathcal{S}$ is graded by the monoid Π_n. Moreover, the maps ψ_τ can be used to introduce a multiplication on $Gr_{\mathbb{K}}(n)$ in the following way:

If $A \in W_\tau$, $B \in W_\sigma$, then define $C = A * B \in W_{\tau\sigma}$ such that $\psi_{\tau\sigma}(A * B) = \psi_\tau(A)\psi_\sigma(B)$

Since multiplication in $\mathcal{S}$ is done by algebraic equations, the set $Gr_{\mathbb{K}}(n)$ viewed as an algebraic variety via the union of the maps ψ_τ, becomes an algebraic variety with a semigroup structure given by polynomial equations, thus, an algebraic semigroup. The structure of algebraic variety of $Gr_{\mathbb{K}}(n)$ obtained this way via the decomposition into affine subspaces $G_{\mathbb{K}}(n) = \bigcup_{\tau \in \Pi_n} W_\tau$ will differ from the one obtained via the Plücker embedding into $\Lambda(\mathbb{K}^n)$.

5.1. **Representations of Π_n and Grassmannian semigroups.** Recall that the set of semistandard Young tableaux admits a semigroup structure called the Plactic monoid, which can be defined in general independently via words in a finite alphabet modulo the Knuth relations. We note now that the set of Young diagrams also have a semigroup structure multiplication. We use the French convention for Young diagrams, with the number of boxes in each row increasing going down.

The shapes of an echelon matrix remind one of Young diagrams. To each type τ we associate a Young diagram $Y_s(\tau)$ in a natural way by placing on each row i of $Y_s(\tau)$ a number k_i of boxes. The Young diagram $Y_s(\tau)$ has strictly increasing number of boxes in its row, and one can associate

a Young diagram $Y(\tau)$ having $k_i - i + 1$ boxes in its i'th row (the number of boxes in the rows of $Y(\tau)$ increase non-strictly going down). Note that τ is completely determined by its Young diagram $Y_s(\tau)$ and $Y(\tau)$. We also note that $Y(\tau)$ is the diagram of a partition of length equal to $k_1 + k_2 - 1 + \cdots + k_t - (t-1)$. Thus $length(Y(\tau)) \leq (n-t+1) + (n-t+2) - 1 + \cdots + (n-t+t) - (t-1) = t(n-t)$. Further, there is a 1-1 bijection between Young diagrams (partitions) with t rows of length at most $t(n-t)$ and the set of types $\tau = (k_1, \ldots, k_t)$ with $k_t \leq n$. Let us also observe that the semigroups Π_n can be embedded in each other via the natural embedding of $M_n(\mathbb{K}) \subset M_{n+1}(\mathbb{K})$ which takes an $n \times n$ matrix and borders it with 0 down and to the right to obtain an $(n+1) \times (n+1)$ matrix. Denote $\Pi = \bigcup_n \Pi_n$; it is a semigroup (but not a monoid) since each successive embedding $\Pi_n \subset \Pi_{n+1}$ is a semigroup map. Moreover, by the above there is a 1-1 bijection between Π and the set $\mathcal{Y}$ of *all* Young diagrams, and also to the set $\mathcal{Y}'$ of all strictly row increasing Young diagrams. Hence, $\mathcal{Y}$ has a semigroup structure introduced by transporting the structure of Π.

More precisely, the $y = (s_1, \ldots, s_t)$ is a partition (Young diagram with rows $s_1 \leq \cdots \leq s_t$), let $T(y) = (s_1, s_2 + 1, \ldots, s_t + t - 1) \in \Pi_n$ is a type for any $n \geq s_t + t - 1$. If y, y' are two Young diagrams, then their multiplication is given by multiplying their associate types $\tau = T(y), \tau' = T(y')$ as elements of the appropriate Π_n and taking Young diagram of the product.

$$y * y' = Y(T(y)T(y'))$$

Multiplication in the semigroup of Young diagrams

The multiplication of the Young diagrams can be described combinatorially as follows. Given Young diagrams $y = (s_1, \ldots, s_t), y' = (l_1, \ldots, l_p)$, first construct the strictly row increasing Young diagrams z, z' by adding $i - 1$ boxes to the i'th non-empty row of y and y' respectively. Let z'' be the Young diagram obtained as follows: count the number s_i of boxes on row i of z, and then the number $m_i = l_{s_i}$ of boxes on row s_i in the second Young diagram z', if that row is non-empty. The number of boxes on row i in z'' is m or empty if the row s_i in z' was empty. Then the product $y * y'$ is obtained by deleting appropriate boxes in z'' to revert it to a non-decreasing Young diagram, i.e. delete 1 box in the 2-nd row of z'', 2 boxes in the third etc. In view of the above connections of Π to Young diagrams and Grassmannians and to better understand Grassmannian semigroups and Π_n, it is interesting to attempt to study their representation theory.

As before, let $\mathcal{S}$ be a Grassmannian semigroup, which we may assume to be in echelon form. Let $\mathbb{F}[\mathcal{S}]$ be its semigroup algebra over some field $\mathbb{F}$. In what follows, we let $R = \mathbb{F}[\mathcal{S}]$ or $R = \mathbb{F}[\Pi_n]$; the results will apply to both semigroup algebras. We denote as before the idempotents $E_i \in \mathcal{S}$ having the first i entries equal to 1 on the main diagonal and 0 elsewhere (they are also elements of Π_n). Denote Z the zero element of $\mathcal{S}$ (in order to distinguish it from the element 0 in $\mathbb{F}[\mathcal{S}]$ and $\mathbb{F}[\mathcal{S}]$). We introduce some notation. In the remaining of this section we will use $+$ and $-$ for the operations of R, that should be distinguished from the analogous operations on matrices inside $M_n(\mathbb{K})$. For each $1 \leq i \leq n$ let $g_i = E_i - E_{i-1} \in R$ (the elements E_i and E_{i-1} are, of course, linearly independent), let $P_i = g_i R$ and $M_i = Span\{(E_i - E_{i-1})A | A \in \mathcal{S}(\text{ or } \Pi_n), E_i A \neq E_i\} \subset P_i$. Let $P_0 = Z \cdot R = \mathbb{K}\{Z\}$. The following is key to the structure of the ring R.

Remark 5.1. *For the duration of this remark only, let us denote $\oplus$ and $\ominus$ the addition and subtraction of matrices in $M_n(\mathbb{K})$ (in order to distinguish these from $+$ and $-$ in R). The product in R of two elements in $\mathcal{S}$ is calculated as the usual product in $M_n(\mathbb{K})$, so there is no danger of confusion there. If $A = (a_{ij})_{i,j} \in \mathcal{S}$ then $g_i A g_j = 0$ whenever $i > j$ or $a_{ij} = 0$. To see*

this, note that we have

$$g_i A g_j = E_i A E_j - E_{i-1} A E_j - E_i A E_{j-1} + E_{i-1} A E_{j-1}$$

We now show that either
(1) $E_i A E_j = E_{i-1} A E_j$ and $E_i A E_{j-1} = E_{i-1} A E_{j-1}$, or
(2) $E_i A E_j = E_i A E_{j-1}$ and $E_{i-1} A E_{j-1} = E_{i-1} A E_j$,
which will prove the claim. These equalities can be regarded as equalities in $M_n(\mathbb{K})$; note that we have assumed $\mathcal{S}$ (or Π_n) is in upper triangular form, so $A = \sum\limits_{k \le l}^{\oplus} a_{kl} e_{kl}$ (meaning a sum in $M_n(\mathbb{K})$), where e_{kl} are the standard matrix basis in $M_n(\mathbb{K})$. It is enough to show that either (1) or (2) holds for $A = a_{kl} e_{kl}$, $k \le l$. Now $a_{kl} E_i e_{kl} E_j = a_{kl} E_{i-1} e_{kl} E_j$ is equivalent to $a_{kl}(E_i \ominus E_{i-1}) e_{kl} E_j = 0$; but $E_i \ominus E_{i-1} = e_{ii}$, so this is further equivalent to $a_{kl} \delta_{ik} e_{il} E_j = 0$. Thus, both equalities in (1) hold if $i \ne k$, or $a_{kl} = 0$. Similarly, if $j \ne l$, one easily sees that both equalities in (2) hold. If $i = k, j = l$, then $i \le j$ the equalities hold since $a_{ij} = 0$ is assumed in this case.

Proposition 5.2. *Let $A \in \mathcal{S}$ be such that $E_i A \ne E_i$. If $i \ge j$, then $g_i A g_j = 0$.*

Proof. By the previous remark, to show the identities (1) and (2) we can only consider the case when $i = k$ and $j = l$. As $i \ge j$ and $k \le l$, we have $i = j$, and in this case the assertion follow from the previous remark if we prove that $a_{ii} = 0$. Indeed, if $a_{ii} \ne 0$, since A is in echelon form, one easily sees that it must have pivots at least on lines $1, 2, \ldots, i$, and so, in fact, $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$, with A_1 a $i \times i$ triangular matrix with 1 on diagonal. Using arguments as before, for example, because the sequence $(A^s)_s$ stabilizes, it follows that $A_1 = I_i$, and this shows that $E_i A = E_i$, a contradiction. $\qquad\square$

The above easy computational observation helps determine the Jacobson radical of R. With the notations above of P_i and M_i, we have:

Proposition 5.3. *Each M_i, $i \ge 1$, is a maximal submodule of P_i, and P_i are projective indecomposable. Moreover, $J(R) = M_1 \oplus \cdots \oplus M_n$, $J(R)^{n+1} = 0$, and $S_i = P_i/M_i$, $i \ge 1$ and $S_0 = P_0$ are, up to isomorphism, the $n + 1$ types of simple right R-modules.*

Proof. Let $M = M_1 \oplus \cdots \oplus M_n$. Note that M, and each M_i, are right ideals of R. For this, it suffices to show that if $E_i A \ne E_i$ with $A \in \mathcal{S}$, then $E_i A B \ne E_i$. This follows immediately if we write the matrices E_i, A, B in block triangular form with two blocks of sizes i and $n - i$ respectively on the diagonal.
We now show that $M^{n+1} = 0$. It is enough to consider an element $x = x_{i_1} \ldots x_{i_n} x_{i_{n+1}}$, with $x_{i_t} \in M_{i_t}$ and show that $x = 0$, since every element in M^{n+1} is a sum of such product elements x. This x is an element of R of the type $g_{i_1} A_1 g_{i_2} A_2 \ldots g_{i_n} A_n g_{i_{n+1}} A_{n+1}$, with $A_t \in \mathcal{S}$, $\forall t$. As the sequence $i_1, i_2, \ldots, i_{n+1}$ is contained in $\{1, 2, \ldots, n\}$, it cannot be strictly increasing, so there is some $i_s \ge i_{s+1}$. By the previous proposition, $g_{i_s} A_s g_{i_{s+1}} = 0$ since $E_{i_s} A_s \ne E_{i_s}$ ($g_{i_s} A_s \in M_{i_s}$), and so such a product equals 0 in R. Hence, M is a nilpotent right ideal, and so it must be contained in $J(R)$. Conversely, note that P_i/M_i is simple (it is 1-dimensional), so M_i is maximal in P_i, and so $J(R) \subseteq M$. Now, one notes without difficulty that $M_i = P_i J(R)$. This ends the proof. $\qquad\square$

Corollary 5.4. *We have $\mathrm{Ext}^1_R(S_j, S_i) = 0$ if $j \ge i \ge 0$ and $\mathrm{Ext}^1(S_i, S_0) = \mathrm{Ext}^1(S_0, S_i) = 0$, so the Ext quiver of R consists of a line with n vertices and an isolated point.*

Proof. This follows immediately since $g_j R g_i = 0$ for $j > i$ and $g_i J g_i = g_i M_i g_i = 0$ so $\operatorname{Hom}_R(P_i, P_j) = 0$ if $j > i$ and $\operatorname{Hom}_R(P_i, P_i) = \mathbb{F}$. The part about the extensions with $S_0 = R \cdot Z = \mathbb{F} - \operatorname{Span}\{Z\}$ is obvious. $\qquad\square$

Hence, the Ext quiver of R is $1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow n \ \sqcup \ 0$, the simple modules are 1 dimensional, and the Jacobson radical is nilpotent.

Now let $R = \mathbb{F}[\Pi_n]$, where Π_n is the semigroup of the 2^n possible types of echelon matrices of size n. We may identify the elements P_τ of Π_n with their type $\tau = (k_1, \ldots, k_t)$. By Remark 5.1, we see that $g_i P_\tau g_j = 0$ if P_τ has 0 at position (i, j). This can be avoided only if $k_i = j$. In this case, note that multiplying P_τ to the left by some E_l retains the first l lines of P_τ and everything else is made 0, and multiplying it by E_p to the right retains the upper left $p \times p$ part of P_τ, and everything else is made 0. It is then not hard to notice that in R we have

$$g_i P_\tau g_j = P_{(k_1, k_2, \ldots, k_i)} - P_{(k_1, k_2, \ldots, k_{i-1})}$$

if $k_i = j$. These elements span $g_i R g_j$ and a basis for $g_i R g_j$ is given by the set $\{P_{(k_1, k_2, \ldots, k_{i-1}, j)} - P_{(k_1, k_2, \ldots, k_{i-1})} | 1 \le k_1 < \cdots < k_{i-1} \le j - 1\}$. Hence, we have that

$$\dim(g_i R g_j) = \binom{i - 1}{j - 1}.$$

Hence, using the fact that $R g_j = \bigoplus_{i \le j} g_i R g_j$ and $g_i R = \bigoplus_{i \le j} g_i R g_j$ and well known combinatorial identities, we obtain the following

Corollary 5.5. *If $R = \mathbb{F}[\Pi_n]$ then*
$\dim(R g_i) = 2^{i-1}$ *if $i \ge 1$ and $\dim(R g_0)$.*
$\dim(g_i R) = \binom{n}{i}$ *if $i \ge 1$ and $\dim(g_0 R) = 1$.*

Using the above, the structure of some of the algebras $\mathbb{F}[\Pi_n]$ for small n can be easily determined. We include the following result without proof, which is left to the reader, but note that (iii) follows from the fact that $\dim(g_i R g_{i+1}) = i = \dim(\operatorname{Ext}^1_R(S_{i+1}, S_i))$.

Corollary 5.6. *(i) $\mathbb{F}[P_2] \cong \mathbb{F} \times T_2(\mathbb{F})$, where $T_2(\mathbb{F})$ is the algebra of upper triangular 2×2 matrices over $\mathbb{F}$.*
(ii) $\mathbb{F}[P_3]$ decomposes into indecomposable projectives of dimensions $4, 2, 1, 1$, as a left module and into indecomposable projectives of dimensions $(1, 3, 3, 1)$ as right modules, and it is isomorphic to path algebra of the quiver
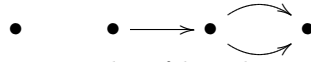
with one relation that identifies the two paths of length 2.
(iii) $\mathbb{F}[P_n]$ is a quotient of the path algebra of the quiver Q_n with $n + 1$ vertices
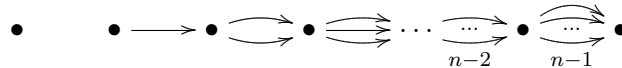
where between vertex i and $i + 1$ there are i arrows (all oriented to the right).

We also remark that $\mathbb{F}[\Pi_n]$ has a bialgebra structure, as a semigroup bialgebra, with comultiplication given by $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$ for $x \in \Pi_n$. Thus, representations of $\mathbb{F}[\Pi_n]$ have a natural tensor product, and the free abelian group on the equivalence classes of representations of Π_n becomes a ring - the representation ring (or Green ring) of $\mathbb{F}[\Pi_n]$.

It would perhaps be interesting to determine the structure of $\mathbb{F}[\Pi_n]$ as a quiver algebra with relations. As we have seen before,

$$g_i P_\tau g_j = P_{(k_1, k_2, \ldots, k_i)} - P_{(k_1, k_2, \ldots, k_{i-1})}$$

and this the elements $R_{k_1, k_2, \ldots, k_i} = \{P_{(k_1, k_2, \ldots, k_{i-1}, j)} - P_{(k_1, k_2, \ldots, k_{i-1})} | 1 \leq k_1 < \cdots < k_{i-1} \leq j-1, k_i = j\}$ provides a basis for $g_i R g_j$. One can show that the multiplication on this basis of R consisting of these elements and including g_0, is done as follows. If $R_{(k(1), \ldots, k(i))} \in g_i R g_j$ is such that $k(i) = j$ and $R_{(s(1), \ldots, s(j))} \in g_j R g_k$ is such that $s(j) = k$, then

$$R_{(k(1), \ldots, k(i))} \cdot R_{(s(1), \ldots, s(j))} = R_{(sk(1), \ldots, sk(i))}; \quad sk(i) = k$$

which makes the "non-zero" part $\bigoplus_{1 \leq i \leq j} g_i R g_j$ of $\mathbb{F}[\Pi_n]$ into a monoid algebra. These elements can be used to be identified with certain paths in the path algebra of the above quiver Q_n.

One may also wonder what is the relation of $\mathbb{F}[\Pi_n]$ and the Grassmann (exterior) algebra $\Lambda_n(\mathbb{F})$. Of course, they are not isomorphic: $\mathbb{F}[\Pi_n]$ has n simple 1-dimensional modules, and is not Frobenius, while $\Lambda_n(\mathbb{F})$ is even a Hopf algebra (so it is Frobenius), and it is local.

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