

Introduction to Financial Derivatives

Understanding the Stock Pricing Model

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Stock Pricing Model

Recall our *stochastic differential equation* to model stock prices:

$$\frac{dS}{S} = \sigma dX + \mu dt$$

where

- μ is known as the asset's *drift*, a measure of the average rate of growth of the asset price,
- σ is the *volatility* of the stock, it measures the standard deviation of an asset's returns, and
- dX is a random sample drawn from a normal distribution with mean zero.

Both μ and σ are measured on a 'per year' basis.

Efficient Market Hypothesis

- Past history is fully reflected in the present price, however this does not hold any further information. (Past performance is not indicative of future returns)
- Markets respond immediately to any new information about an asset.

Markov Process

Definition

A stochastic process where only the present value of a variable is relevant for predicting the future.

This implies that knowledge of the past history of a Markov variable is irrelevant for determining future outcomes.

Markov Process \Leftrightarrow Efficient Market Hypothesis

Investigating the Random Variable

Consider a random variable, X , that follows a Markov stochastic process.

Further assume that the variable's change (over a one-year time span), dX , can be characterized by a standard normal distribution (a probability distribution with mean zero and standard deviation one, $\phi = \varphi(0, 1)$).

What is the probability distribution of the change in the value of the variable (dX) over two years?

Investigating the Random Variable

Since X follows a Markov process, the two probability distributions are independent. Thus, we can sum the distributions.

The two year mean is the sum of the two one-year means.

Similarly, the two year variance is the sum of the two one-year variances.

However, the change is best represented by the standard deviation, so the probability distribution that describes dX over two years is: $\varphi(0, \sqrt{2})$.

Investigating the Random Variable

Assumption

Changes in variance are equal for all identical time intervals.

- For a six month period, the variance of change is 0.5 and the standard deviation of the change is $\sqrt{0.5}$. The probability distribution for the change in the value of the variable during six months is $\phi(0, \sqrt{0.5})$.
- Similarly, dX over a three month period is $\phi(0, \sqrt{0.25})$.
- The change in the value of the variable during any time period, dt , is $\phi(0, \sqrt{dt}) \Leftrightarrow \phi \sqrt{dt}$.

This is because the variance of the changes in successive time periods are additive, while the standard deviations are not.

Wiener Process

The process followed by the variable we have been considering is known as a *Wiener process*; A particular type of Markov stochastic process with a mean change of zero and a variance rate of 1 per year.

- The change, dX during a small period of time, dt , is

$$dX = \phi \sqrt{dt}$$

where $\phi = \phi(0, 1)$ as defined above.

- The values of dX for any two different short intervals of time, dt , are independent.

Fact

In physics the Wiener process is referred to as Brownian motion and is used to describe the random movement of particles.

Wiener Statistics

- Mean of dX , $\mathbb{E}[dX] = \sqrt{dt}\mathbb{E}[\phi] = 0$
- Variance of dX ,
 $\text{Var}[dX] = \mathbb{E}[(dX - 0)^2] = \mathbb{E}[\phi^2 dt] = dt\mathbb{E}[\phi^2] = dt \cdot 1 = dt$
- Standard deviation of $dX = \sqrt{dt}$

The Pricing Model

$$\frac{dS}{S} = \sigma dX + \mu dt$$

- Since we chose dX such that $\mathbb{E}[dX] = 0$ the mean of dS is:

$$\mathbb{E}[dS] = \mathbb{E}[\sigma S dX + \mu S dt] = \mu S dt$$

- The variance of dS is:

$$\text{Var}[dS] = \mathbb{E}[dS^2] - \mathbb{E}[dS]^2 = \mathbb{E}[\sigma^2 S^2 dX^2] = \sigma^2 S^2 dt$$

Note that the standard deviation equals $\sigma S \sqrt{dt}$, which is proportional to the asset's volatility.

Taylor's +

We need to determine how to calculate small changes in a function that is dependent on the values determined by the above stochastic differential equation. Let $f(S)$ be the desired smooth function of S ; since f is sufficiently smooth we know that small changes in the asset's price, dS , result in small changes to the function f . Recall that we approximated df with a Taylor series expansion, resulting in

$$df = \frac{df}{dS} dS + \frac{1}{2} \frac{d^2 f}{dS^2} dS^2 + \dots,$$

where $dS = \sigma S dX + \mu S dt \implies$

$$dS^2 = (\sigma S dX + \mu S dt)^2 = \sigma^2 S^2 dX^2 + 2\sigma\mu S^2 dt dX + \mu^2 S^2 dt^2$$

$dX^2 \rightarrow ?$ as $dt \rightarrow 0$

Assumption

As $dt \rightarrow 0$, $dX = \mathcal{O}(\sqrt{dt}) \Leftrightarrow dX/\sqrt{dt} = 1$ and
 $dXdX = o(dt) \Leftrightarrow dXdX = 0$

Implies that

$$dS^2 \rightarrow \sigma^2 S^2 dt \text{ as } dt \rightarrow 0$$

and results in

$$df = \frac{df}{dS}(\sigma S dX + \mu S dt) + \frac{1}{2} \sigma^2 S^2 \frac{d^2 f}{dS^2} dt$$

$$= \sigma S \frac{df}{dS} dX + \left(\mu S \frac{df}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 f}{dS^2} \right) dt$$

$$dX^2 \rightarrow ? \text{ as } dt \rightarrow 0$$

The integrated form of our stochastic differential equation to model stock prices is

$$S(t) = S(t_0) + \sigma \int_{t_0}^t S dX + \mu \int_{t_0}^t S dt$$

but how to handle $\int_{t_0}^t S dX$?

Stochastic Calculus

For any function f ,

$$\int_{t_0}^t f(\tau) dX(\tau) = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} f(t_k)(X(t_{k+1}) - X(t_k))$$

where $t_0 < t_1 < \dots < t_n = t$ is any partition (or division) of the range $[t_0, t]$ into n smaller regions and X is the running sum of the random variables dX .

Note

The value of the function, f , inside the summation is taken at the left-hand end of the small regions (at $t = t_k$ and not at t_{k+1}) – effectively, this is where the Markov Property is incorporated into the model!

Stochastic Calculus

If $X(t)$ were a smooth function the integral would be the usual Stieltjes integral and it would not matter that f was evaluated at the left-hand end. However, because of the randomness (which does not go away as $dt \rightarrow 0$) the fact that the summation depends on the left-hand value of f in each partition becomes important.

Example

$$\int_{t_0}^t X(\tau) dX(\tau) = \frac{1}{2}(X(t)^2 - X(t_0)^2) - \frac{1}{2}(t - t_0)$$

If X were smooth the last term would not be present.

$$dX^2 \rightarrow dt \text{ as } dt \rightarrow 0$$

It can be shown (using stochastic integration) that

$$f(S(t)) = f(S(t_0)) + \int_{t_0}^t \sigma S \frac{df}{dS} dX + \int_{t_0}^t \left(\mu S \frac{df}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 f}{dS^2} \right) dt$$

which when written in shorthand notation becomes

$$df = \sigma S \frac{df}{dS} dX + \left(\mu S \frac{df}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 f}{dS^2} \right) dt$$

Further Generalization

Now consider f to be a function of both S and t . So long as we are aware of partial derivatives, we can once again expand our function (now $f(S + dS, t + dt)$) using a Taylor series approximation about (S, t) to get:

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2 + \dots,$$

substituting in our past work, we end up with the following result:

$$df = \sigma S \frac{\partial f}{\partial S} dX + \left(\mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt$$

Assumptions

- The asset price follows a lognormal random walk
- The risk-free interest rate r and the volatility of the underlying asset σ are known functions of time over the life of the option.
- There are no associated transaction costs.
- The underlying asset pays no dividends during the life of the option.
- There are no arbitrage opportunities.
- Trading of the underlying asset can take place continuously.
- Short selling is allowed (full use of proceeds from the sale is permitted)
- fractional shares of the underlying asset may be traded.

Another Riskless Portfolio

Construct a portfolio, Π_2 whose variation over a small time period, dt is wholly deterministic.

Let

$$\Pi_2 = -f + \Delta S \quad (1)$$

our portfolio is short one derivative security (we don't know or care if it's a call or put) and long Δ of the underlying stock. Δ is a given number whose value (while not yet determined) is constant throughout each time step.

Another Riskless Portfolio

We are interested in how our portfolio reacts to small variations.
 We observe that

$$\begin{aligned}
 d\Pi_2 &= -df + \Delta dS \\
 &= -\sigma S \frac{\partial f}{\partial S} dX - \left(\mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt + \Delta (\sigma S dX + \mu S dt) \\
 &= -\sigma S \left(\frac{\partial f}{\partial S} - \Delta \right) dX - \left(\mu S \left(\frac{\partial f}{\partial S} - \Delta \right) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt
 \end{aligned}$$

Choice of Delta

Choosing $\Delta = \frac{\partial f}{\partial S}$ we have:

$$d\Pi_2 = -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt \quad (2)$$

this equation has no dependence on dX and therefore must be riskless during time dt . Furthermore since we have assumed that arbitrage opportunities do not exist, Π_2 must earn the same rate of return as other short-term risk-free securities over the short time period we defined by dt . It follows that

$$d\Pi_2 = r\Pi_2 dt$$

where r is the risk-free interest rate.

Black-Scholes

Substituting the different values of Π_2 into the above equation we have

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt = r(f - \frac{\partial f}{\partial S}S)dt$$

which when simplified gives us

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \quad (3)$$

the *Black-Scholes partial differential equation*. Under the stated assumptions any derivative security whose value depends only on the current value of the underlying asset S and on time t *must* satisfy the above equation.

Not just B-S

- The Black-Scholes equation has many different solutions; the particular derivative that is obtained when the equation is solved depends on the boundary conditions that are used. For example if the derivative in question is a European call option then the key associated boundary condition will be:

$$f = \max(S - E, 0) \quad \text{when } t = T$$

- Equation (3) is not riskless for all time—it is only riskless for the amount of time specified by dt . This is because as S and t change so does $\Delta = \frac{\partial f}{\partial S}$, thus to keep the portfolio defined by Π_2 riskless we need to constantly update number of shares of underlying held.

Black-Scholes Solved

Consider the Black-Scholes equation (and boundary conditions) for a European call with value $C(S, t)$

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0$$

with

$$C(0, t) = 0, \quad \text{and} \quad C(S, t) \sim S \quad \text{as } S \rightarrow \infty$$

and

$$C(S, T) = \max(S - E, 0)$$

- Notice the similarities to the one-dimensional diffusion equation; how can we use this observation?

Substitutions

We need to get rid of the ugly S and S^2 terms in the equation above, so we make the following substitutions:

- $S = Ee^x$
- $t = T - \tau/\frac{1}{2}\sigma^2$
- $C = Ev(x, \tau)$

Substitutions

The above substitutions result in the following equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k - 1) \frac{\partial v}{\partial x} - kv$$

where

$$k = r / \frac{1}{2} \sigma^2$$

and the initial condition becomes

$$v(x, 0) = \max(e^x - 1, 0)$$

Closer

Note the above equation contains only one dimensionless parameter, k , and is almost the diffusion equation. Consider the following change of variable

$$v = e^{\alpha x + \beta \tau} u(x, \tau)$$

for some constants α and β to be determined later. Making the substitution (and performing the differentiation) results in

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku$$

now if we choose $\beta = \alpha^2 + (k-1)\alpha - k$ with $0 = 2\alpha + (k-1)$ we return an equation with no u term and no $\frac{\partial u}{\partial x}$ term.

NEED TO DO MORE!!!