CONVERGENCE OF THE GENERALIZED-α METHOD FOR CONSTRAINED SYSTEMS IN MECHANICS WITH NONHOLONOMIC CONSTRAINTS

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Abstract. We present a convergence analysis for a second order extension of the generalized-α method of Chung and Hulbert for systems in mechanics having nonconstant mass matrix and nonholonomic constraints.

Key words. Differential-algebraic equations, generalized-α method, nonholonomic constraints.

AMS subject classifications. 65L05, 65L06, 65L80, 70F20, 70H03, 70H05, 70H45.

1. Introduction. The generalized-α method of Chung and Hulbert [2] was originally developed for second order systems of differential equations in structural dynamics of the form $M\ddot{y} = f(t, y, \dot{y})$. In mechanics $M \in \mathbb{R}^{n\times n}$ is a constant mass matrix, $y \in \mathbb{R}^n$ is a vector of generalized coordinates, $\dot{y} \in \mathbb{R}^n$ is a vector of generalized velocities, $\ddot{y} \in \mathbb{R}^n$ is a vector of generalized accelerations, and $f(t, y, \dot{y}) \in \mathbb{R}^n$ represents forces. Introducing the new variables $z := \dot{y} \in \mathbb{R}^n$ and $a := \dot{z} = \ddot{y} \in \mathbb{R}^n$, these equations are equivalent to the semi-explicit system of differential-algebraic equations (DAEs)

$$\dot{y} = z, \quad \dot{z} = a, \quad 0 = Ma - f(t, y, z). \tag{1.1}$$

Assuming M to be nonsingular, this system of DAEs is of index 1 since one can obtain explicitly $a = M^{-1}f(t, y, z)$. The generalized-α method of Chung and Hulbert [2] for $M\ddot{y} = f(t, y, \dot{y})$ or equivalently for (1.1) is a non-standard implicit one-step method. One step of this method $(t_0, y_0, z_0, a_\alpha) \mapsto (t_1 = t_0 + h, y_1, z_1, a_{1+\alpha})$ with step-size h can be expressed as follows

$$y_1 = y_0 + hz_0 + \frac{h^2}{2}\left((1-2\beta)a_\alpha + 2\beta a_{1+\alpha}\right), \tag{1.2a}$$

$$z_1 = z_0 + h\left((1-\gamma)a_\alpha + \gamma a_{1+\alpha}\right), \tag{1.2b}$$

$$(1-\alpha_m)Ma_{1+\alpha} + \alpha_m M a_\alpha = (1-\alpha_f)f(t_1, y_1, z_1) + \alpha_f f(t_0, y_0, z_0). \tag{1.2c}$$

The generalized-α method has free coefficients $\alpha_m \neq 1, \alpha_f, \beta, \gamma$. Let $\alpha := \alpha_m - \alpha_f$, $t_\alpha := t_0 + \alpha h$, and $t_{1+\alpha} := t_0 + (1+\alpha)h$. A justification of the notation $a_\alpha, a_{1+\alpha}$ comes from the fact that for a solution $(y(t), z(t), a(t))$ and values (y_0, z_0, a_α) satisfying $y_0 - y(t_0) = O(h^2)$, $z_0 - z(t_0) = O(h^2)$, $a_\alpha - a(t_\alpha) = O(h^2)$, we have $a_{1+\alpha} - a(t_{1+\alpha}) = O(h^2)$, whereas we only have $a_{1+\alpha} - a(t_1) = O(h)$ when $\alpha \neq 0$, see [5]. For specific choices of the generalized-α coefficients we obtain well-known methods:

- Newmark's family: $\alpha_m = 0$, $\alpha_f = 0$;
  - The trapezoidal rule: $\beta = 1/4$, $\gamma = 1/2$;
  - Störmer's rule: $\beta = 0$, $\gamma = 1/2$;
- The Hilber-Hughes-Taylor α (HHT-α) method [3, 4]:

$$\alpha_m = 0, \quad \alpha := -\alpha_f \in \left[-\frac{1}{3}, 0\right], \quad \beta = \frac{(1-\alpha)^2}{4}, \quad \gamma = \frac{1}{2} - \alpha.$$

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The coefficients $\alpha_m \neq 1, \alpha_f, \beta, \gamma$ of the generalized-α method (1.2) are usually chosen according to

$$\alpha_m = \frac{2\rho_\infty - 1}{1+\rho_\infty}, \qquad \alpha_f = \frac{\rho_\infty}{1+\rho_\infty}, \qquad \beta = \frac{(1-\alpha)^2}{4}, \qquad \gamma = \frac{1}{2} - \alpha$$

where $\rho_\infty \in [0,1]$ is a parameter controlling numerical dissipation ($\rho_\infty = 0$ for maximal dissipation [2]).

In this paper we analyze an extension of the generalized-α method (1.2) for systems having possibly a nonconstant mass matrix $M = M(t,y)$ and nonholonomic constraints $k(t,y,\dot{y}) = 0$. Such systems are frequently encountered in multibody dynamics [6]. Our analysis here is based on the extension proposed by L. O. Jay and D. Negrut in [5]. In section 2 the extended generalized-α method is presented. In section 3 we give an existence and uniqueness result for this method. In section 4 the local error of this method is analyzed. In section 5 we analyze the propagation of perturbations by considering two neighbouring generalized-α solutions. In section 6 convergence of the generalized-α method is given and global error estimates are obtained showing second order of convergence. Some numerical experiments are reported in Section 7. A short conclusion is finally given in Section 8.

2. Extension of the generalized-α method for systems with nonholonomic constraints. We consider systems having a nonconstant mass matrix $M = M(t,y)$ and nonholonomic constraints $k(t,y,\dot{y}) = 0$, more precisely

$$M(t,y)\ddot{y} = f(t,y,\dot{y},\psi), \quad 0 = k(t,y,\dot{y}).$$

Usually $f(t,y,\dot{y},\psi) = f_0(t,y,\dot{y}) - k_{\dot{y}}^T(t,y,\dot{y})\psi$ and the term $-k_{\dot{y}}^T(t,y,\dot{y})\psi$ containing algebraic variables ψ represents reaction forces due to the nonholonomic constraints $k(t,y,\dot{y}) = 0$. Hence, we consider systems of index 2 DAEs of the form

$$\dot{y} = z, \quad \dot{z} = a, \quad 0 = M(t,y)a - f(t,y,z,\psi), \tag{2.1a}$$

$$0 = k(t,y,z), \tag{2.1b}$$

and we assume the matrix

$$\begin{pmatrix} M(t,y) & -f_\psi(t,y,z,\psi) \\ k_z(t,y,z) & O \end{pmatrix} \text{ is nonsingular .} \tag{2.1c}$$

When $f(t,y,z,\psi) = f_0(t,y,z) - k_z^T(t,y,z)\psi$, this matrix becomes

$$\begin{pmatrix} M(t,y) & k_z^T(t,y,z) \\ k_z(t,y,z) & O \end{pmatrix}$$

and it is symmetric when $M(t,y)$ is symmetric. At t_0 we consider consistent initial conditions (y_0, z_0), i.e.,

$$0 = k(t_0, y_0, z_0), \tag{2.2a}$$

and also (a_0, ψ_0) satisfying

$$0 = M(t_0,y_0)a_0 - f(t_0,y_0,z_0,\psi_0), \tag{2.2b}$$

$$0 = k_t(t_0,y_0,z_0) + k_y(t_0,y_0,z_0)z_0 + k_z(t_0,y_0,z_0)a_0. \tag{2.2c}$$

The generalized-α method (1.2) can be extended to systems having nonholonomic constraints as follows

$$y_1 = y_0 + hz_0 + \frac{h^2}{2}\left((1-2\beta)a_\alpha + 2\beta a_{1+\alpha}\right), \tag{2.3a}$$
$$z_1 = z_0 + h\left((1-\gamma)a_\alpha + \gamma a_{1+\alpha}\right), \tag{2.3b}$$
$$(1-\alpha_m)M_{1+\alpha}a_{1+\alpha} + \alpha_m M_\alpha a_\alpha = (1-\alpha_f)f(t_1,y_1,z_1,\psi_1) + \alpha_f f(t_0,y_0,z_0,\psi_0), \tag{2.3c}$$
$$0 = k(t_1,y_1,z_1) \tag{2.3d}$$

where $a_\alpha \approx a(t_\alpha)$, and

$$M_{1+\alpha} \approx M(t_{1+\alpha}, y(t_{1+\alpha})), \quad M_\alpha \approx M(t_\alpha, y(t_\alpha)),$$

for example we can take explicitly

$$M_{1+\alpha} := M(t_{1+\alpha}, y_0 + h(1+\alpha)z_0), \quad M_\alpha := M(t_\alpha, y_0 + h\alpha z_0) \ \text{ or } \ M_{(1+\alpha)-1}$$

where $M_{(1+\alpha)-1}$ denotes the matrix $M_{1+\alpha}$ used at the previous time-step.

3. Existence and uniqueness of the generalized-α solution.

THEOREM 3.1. *Consider the system of index 2 DAEs (2.1) with initial conditions $(y_0, z_0, \psi_0) = (y_0(h), z_0(h), \psi_0(h))$ at t_0 and $a_\alpha = a_\alpha(h)$ at $t_\alpha := t_0 + \alpha h$ all depending on h satisfying for $h \to 0$*

$$\begin{aligned}
&k(t_0,y_0,z_0) = o(h),\\
&M(t_0,y_0)a_\alpha - f(t_0,y_0,z_0,\psi_0) = o(1),\\
&k_t(t_0,y_0,z_0) + k_y(t_0,y_0,z_0)z_0 + k_z(t_0,y_0,z_0)a_\alpha = o(1).
\end{aligned}$$

Suppose $\gamma \neq 0$, $\alpha_m \neq 1$, and $\alpha_f \neq 1$. Then there exists $h_{\max} > 0$ such that for h satisfying $|h| \leq h_{\max}$ there exists a unique solution $(y_1, z_1, a_{1+\alpha}, \psi_1)$ depending on h to the system of equations (2.3) in a neighborhood of $(y_0, z_0, a_\alpha, \psi_0)$. Moreover, for $h \to 0$ we have the estimates

$$y_1 - y_0 = O(h), \quad z_1 - z_0 = O(h), \quad a_{1+\alpha} - a_\alpha = O(h), \quad \psi_1 - \psi_0 = O(h). \tag{3.1}$$

Proof. In (2.3d) replacing y_1 and z_1 using (2.3a) and (2.3b) respectively and then multiplying (2.3d) by $1/h$, the system of nonlinear equations (2.3) can be rewritten equivalently as

$$y_1 - \left(y_0 + hz_0 + \frac{h^2}{2}\left((1-2\beta)a_\alpha + 2\beta a_{1+\alpha}\right)\right) = 0, \tag{3.2a}$$
$$z_1 - \left(z_0 + h\left((1-\gamma)a_\alpha + \gamma a_{1+\alpha}\right)\right) = 0, \tag{3.2b}$$
$$(1-\alpha_m)M_{1+\alpha}a_{1+\alpha} + \alpha_m M_\alpha a_\alpha - \left((1-\alpha_f)f(t_1,y_1,z_1,\psi_1) + \alpha_f f(t_0,y_0,z_0,\psi_0)\right) = 0, \tag{3.2c}$$
$$\frac{1}{h}k\left(t_1, y_0 + hz_0 + \frac{h^2}{2}\left((1-2\beta)a_\alpha + 2\beta a_{1+\alpha}\right), z_0 + h\left((1-\gamma)a_\alpha + \gamma a_{1+\alpha}\right)\right) = 0. \tag{3.2d}$$

Expanding the function k in (3.2d) around the argument (t_0, y_0, z_0) we obtain

$$\begin{aligned} 0 = & \frac{1}{h} k(t_0, y_0, z_0) \\ & + k_t(t_0, y_0, z_0) + k_y(t_0, y_0, z_0) z_0 + k_z(t_0, y_0, z_0) \left((1-\gamma) a_\alpha + \gamma a_{1+\alpha}\right) + O(h). \end{aligned}$$

Hence, by using the hypotheses of the theorem the system of equations (3.2) is satisfied at $h = 0$ by

$$(y_1(0), z_1(0), a_{1+\alpha}(0), \psi_1(0)) = (y_0(0), z_0(0), a_\alpha(0), \psi_0(0)).$$

The Jacobian with respect to $(y_1, z_1, a_{1+\alpha}, \psi_1)$ at $h = 0$ of the system of equations (3.2) is of the form

$$\begin{pmatrix} I & O & O & O \\ O & I & O & O \\ * & * & (1-\alpha_m) M_0 & -(1-\alpha_f) f_{\psi,0} \\ O & O & \gamma k_{z,0} & O \end{pmatrix}$$

where

$$M_0 := M(t_0, y_0(0)), \quad f_{\psi,0} := f_\psi(t_0, y_0(0), z_0(0), \psi_0(0)), \quad k_{z,0} := k_z(t_0, y_0(0), z_0(0)).$$

This Jacobian matrix is nonsingular since the block matrix

$$\begin{aligned} & \begin{pmatrix} (1-\alpha_m) M_0 & -(1-\alpha_f) f_{\psi,0} \\ \gamma k_{z,0} & O \end{pmatrix} \\ & \qquad = \begin{pmatrix} (1-\alpha_m) I & O \\ O & \gamma I \end{pmatrix} \begin{pmatrix} M_0 & -f_{\psi,0} \\ k_{z,0} & O \end{pmatrix} \begin{pmatrix} I & O \\ O & \frac{(1-\alpha_f)}{(1-\alpha_m)} I \end{pmatrix} \end{aligned}$$

is nonsingular under the assumptions $\gamma \neq 0$, $\alpha_m \neq 1$, and $\alpha_f \neq 1$. The conclusion and the estimates (3.1) now follow from the application of the implicit function theorem. □

4. Local error. We now consider local error estimates:

THEOREM 4.1. *Consider the system of index 2 DAEs (2.1) with consistent initial conditions* (y_0, z_0, a_0, ψ_0) *at* t_0*, see (2.2), and exact solution* $(y(t), z(t), a(t), \psi(t))$*. Suppose* $\gamma \neq 0$*,* $\alpha_m \neq 1$*, and* $\alpha_f \neq 1$*. Consider* a_α *satisfying for* $h \to 0$

$$a_\alpha - a(t_0 + \alpha h) = O(h^q) \quad q = 1 \quad \text{or} \quad 2.$$

For $q = 2$ *we suppose in addition that* $\gamma = \frac{1}{2} - \alpha$*. Then for* $|h| \leq h_{\max}$ *the numerical solution* (y_1, z_1, ψ_1) *at* $t_1 := t_0 + h$ *and* $a_{1+\alpha}$ *at* $t_{1+\alpha} := t_0 + (1+\alpha)h$ *to the system of equations (2.3) satisfies*

(4.1a) $\quad y_1 - y(t_1) = O(h^3), \quad z_1 - z(t_1) = O(h^{q+1}), \quad a_{1+\alpha} - a(t_{1+\alpha}) = O(h^q),$

(4.1b) $\quad \psi_1 - \psi(t_1) = O(h^q).$

Proof. The Taylor series at t_0 of the exact solution $(y(t), z(t))$ satisfies

$$\begin{aligned} y(t_0 + h) &= y_0 + h z_0 + \frac{h^2}{2} a_0 + O(h^3), \\ z(t_0 + h) &= z_0 + h a_0 + \frac{h^2}{2} \dot{a}_0 + O(h^3) \end{aligned}$$

where $\dot a_0 := \dot a(t_0)$. We need $\dot a$ and $\dot\psi$ from the exact solution. They can be obtained from the equations

$$\begin{aligned}
0 &= \frac{d}{dt}\left(M(t,y)a - f(t,y,z,\psi)\right)\\
&= M_t(t,y)a + M_y(t,y)(z,a) + M(t,y)\dot a \qquad (4.2a)\\
&\quad -f_t(t,y,z,\psi) - f_y(t,y,z,\psi)z - f_z(t,y,z,\psi)a - f_\psi(t,y,z,\psi)\dot\psi,\\
0 &= \frac{d^2}{dt^2}k(t,y,z) = \frac{d}{dt}\left(k_t(t,y,z) + k_y(t,y,z)z + k_z(t,y,z)a\right)\\
&= k_{tt}(t,y,z) + 2k_{ty}(t,y,z)z + 2k_{tz}(t,y,z)a + k_{yy}(t,y,z)(z,z) \qquad (4.2b)\\
&\quad +2k_{yz}(t,y,z)(z,a) + k_y(t,y,z)a + k_{zz}(t,y,z)(a,a) + k_z(t,y,z)\dot a
\end{aligned}$$

which can be expressed as a system of linear equations for $\dot a$ and $\dot\psi$

$$\begin{pmatrix} M(t,y) & -f_\psi(t,y,z,\psi) \\ k_z(t,y,z) & O \end{pmatrix}\begin{pmatrix}\dot a\\ \dot\psi\end{pmatrix} \qquad (4.3)$$
$$= -\begin{pmatrix} M_t(t,y)a + M_y(t,y)(z,a) - f_t(t,y,z,\psi) - f_y(t,y,z,\psi)z - f_z(t,y,z,\psi)a \\ k_{tt}(t,y,z) + 2k_{ty}(t,y,z)z + \ldots + k_y(t,y,z)a + k_{zz}(t,y,z)(a,a)\end{pmatrix}.$$

The values of $\dot a_0 := \dot a(t_0)$ and $\dot\psi_0 := \dot\psi(t_0)$ can be obtained from these linear equations by taking the arguments $t_0, y_0, z_0, a_0, \psi_0$ in the functions above. From $a_\alpha - a(t_0+\alpha h) = O(h)$ we get $a_\alpha - a_0 = O(h)$. Hence, from $a_{1+\alpha} - a_\alpha = O(h)$ of the existence and uniqueness Theorem 3.1 we have $a_{1+\alpha} - a_0 = O(h)$. Therefore we obtain directly from (2.3a) and (2.3b)

$$y_1 = y_0 + hz_0 + \frac{h^2}{2}a_0 + O(h^3), \quad z_1 = z_0 + ha_0 + O(h^2),$$

and thus

$$y_1 - y(t_0+h) = O(h^3), \quad z_1 - z(t_0+h) = O(h^2).$$

Hence, we have proved (4.1) for $q = 1$.

Now suppose that $q = 2$, i.e., that

$$a_\alpha = a_0 + h\alpha\dot a_0 + O(h^2).$$

We will show that

$$a'_{1+\alpha}(0) := \frac{d}{dh}a_{1+\alpha}(h)\bigg|_{h=0}, \quad \psi'_1(0) := \frac{d}{dh}\psi_1(h)\bigg|_{h=0}$$

satisfy

$$a'_{1+\alpha}(0) = (1+\alpha)\dot a_0, \quad \psi'_1(0) = \dot\psi_0.$$

This will be proved by showing that $\frac{1}{(1+\alpha)}a'_{1+\alpha}(0)$ and $\psi'_1(0)$ satisfy the system of linear equations (4.3) with function arguments $t_0, y_0, z_0, a_0, \psi_0$. For that purpose, we consider first

$$0 = \frac{1}{h^2}k(t_1, y_1, z_1) = \frac{1}{h^2}k(t_0+h, y_0+\delta_y, z_0+\delta_z)$$

where from (2.3a)-(2.3b) we have

$$\delta_y := hz_0 + \frac{h^2}{2}\left((1-2\beta)a_\alpha + 2\beta a_{1+\alpha}\right), \quad \delta_z := h\left((1-\gamma)a_\alpha + \gamma a_{1+\alpha}\right).$$

Expanding $k(t_0+h, y_0+\delta_y, z_0+\delta_z)$ in Taylor series around (t_0, y_0, z_0) we get

$$\begin{aligned} k(t_0+h, y_0+\delta_y, z_0+\delta_z) = {} & k_0 + hk_{t0} + k_{y0}\delta_y + k_{z0}\delta_z + \frac{1}{2}h^2 k_{tt0} + hk_{ty0}\delta_y \\ & + hk_{tz0}\delta_z + \frac{1}{2}k_{yy0}(\delta_y,\delta_y) + \frac{1}{2}k_{yz0}(\delta_y,\delta_z) + \frac{1}{2}k_{zz0}(\delta_z,\delta_z) \\ & + O(h^3 + \|\delta_y\|^3 + \|\delta_z\|^3), \end{aligned}$$

and

$$\begin{aligned} \delta_y &= hz_0 + \frac{h^2}{2}a_0 + \frac{h^3}{2}\left((1-2\beta)\alpha\dot{a}_0 + 2\beta a'_{1+\alpha}(0)\right) + O(h^4), \\ \delta_z &= ha_0 + h^2\left((1-\gamma)\alpha\dot{a}_0 + \gamma a'_{1+\alpha}(0)\right) + O(h^3). \end{aligned}$$

We have

$$0 = \lim_{h\to 0}\frac{1}{h^2}k(t_1, y_1, z_1) = \lim_{h\to 0}\frac{1}{h^2}k(t_0+h, y_0+\delta_y, z_0+\delta_z)$$

which from the results above leads to

$$\begin{aligned} 0 = {} & k_{y0}a_0 + 2k_{z0}\left((1-\gamma)\alpha\dot{a}_0 + \gamma a'_{1+\alpha}(0)\right) + k_{tt0} + 2k_{ty0}z_0 + 2k_{tz0}a_0 + k_{yy0}(z_0,z_0) \\ & + 2k_{yz0}(z_0,a_0) + k_{zz0}(a_0,a_0), \end{aligned}$$

i.e.,

$$\begin{aligned} 0 = {} & k_{tt0} + 2k_{ty0}z_0 + 2k_{tz0}a_0 + k_{yy0}(z_0,z_0) + 2k_{yz0}(z_0,a_0) + k_{y0}a_0 + k_{zz0}(a_0,a_0) \\ & + k_{z0}\left(2(1-\gamma)\alpha\dot{a}_0 + 2\gamma(1+\alpha)\frac{1}{(1+\alpha)}a'_{1+\alpha}(0)\right), \end{aligned} \tag{4.4}$$

compare with (4.2b). Notice that

$$2(1-\gamma)\alpha + 2\gamma(1+\alpha) = 2(\alpha+\gamma) = 2\cdot\frac{1}{2} = 1.$$

On the other hand from (3.2c) we get

$$\begin{aligned} 0 = {} & (1-\alpha_m)\left(M_0 + h(1+\alpha)M_{t0} + h(1+\alpha)M_{y0}(z_0,\cdot)\right)\left(a_{1+\alpha}(0) + ha'_{1+\alpha}(0)\right) + O(h^2) \\ & + \alpha_m\left(M_0 + h\alpha M_{t0} + h\alpha M_{y0}(z_0,\cdot)\right)\left(a_0 + h\alpha\dot{a}_0\right) + O(h^2) \\ & - (1-\alpha_f)\Big(f_0 + hf_{t0} + hf_{y0}z_0 + hf_{z0}a_0 + hf_{\psi 0}\psi'_1(0)\Big) + O(h^2) - \alpha_f f_0 \\ = {} & \underbrace{M_0 a_0 - f_0}_{=0} \\ & + h\Big((1-\alpha_m)(1+\alpha)\left(M_{t0}a_0 + M_{y0}(z_0,a_0)\right) + (1-\alpha_m)M_0(1+\alpha)\frac{1}{(1+\alpha)}a'_{1+\alpha}(0) \\ & \quad + \alpha_m\alpha\left(M_{t0}a_0 + M_{y0}(z_0,a_0)\right) + \alpha_m\alpha M_0\dot{a}_0 \\ & \quad - (1-\alpha_f)(f_{t0} + f_{y0}z_0 + f_{z0}a_0 + hf_{\psi 0}\psi'_1(0))\Big) + O(h^2). \end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\Big((1-\alpha_m)M_{1+\alpha}a_{1+\alpha}+\alpha_m M_\alpha a_\alpha-((1-\alpha_f)f(t_1,y_1,z_1,\psi_1)+\alpha_f f_0)\Big)\\
&=\underbrace{(1-\alpha_m+\alpha)}_{=1-\alpha_f}(M_{t0}a_0+M_{y0}(z_0,a_0))\\
&\quad+(1-\alpha_m)(1+\alpha)M_0\frac{1}{(1+\alpha)}a'_{1+\alpha}(0)+\alpha_m\alpha M_0\dot a_0\\
&\quad-(1-\alpha_f)\left(f_{t0}+f_{y0}z_0+f_{z0}a_0+f_{\psi 0}\psi_1'(0)\right)\\
&=(1-\alpha_f)\Bigg(M_{t0}a_0+M_{y0}(z_0,a_0)+M_0\left(\left(1-\frac{\alpha_m\alpha}{(1-\alpha_f)}\right)\frac{1}{(1+\alpha)}a'_{1+\alpha}(0)+\frac{\alpha_m\alpha}{(1-\alpha_f)}\dot a_0\right)\\
&\qquad\qquad-\left(f_{t0}+f_{y0}z_0+f_{z0}a_0+f_{\psi 0}\psi_1'(0)\right)\Bigg).
\end{aligned}$$

Hence, we have obtained

$$\begin{aligned}
0=\;&M_0\left(\frac{\alpha_m\alpha}{(1-\alpha_f)}\dot a_0+\left(1-\frac{\alpha_m\alpha}{(1-\alpha_f)}\right)\frac{1}{(1+\alpha)}a'_{1+\alpha}(0)\right)+M_{t0}a_0+M_{y0}(z_0,a_0)\\
&-\left(f_{t0}+f_{y0}z_0+f_{z0}a_0+f_{\psi 0}\psi_1'(0)\right). \qquad (4.5)
\end{aligned}$$

From (4.3) the equations (4.4) and (4.5) are satisfied when replacing $\frac{1}{(1+\alpha)}a'_{1+\alpha}(0)$ by $\dot a_0$ and $\psi_1'(0))$ by $\dot\psi_0$. Hence, by uniqueness of a solution we must have

$$\frac{1}{(1+\alpha)}a'_{1+\alpha}(0)=\dot a_0,\quad \psi_1'(0)=\dot\psi_0.$$

From this result we obtain

$$\begin{aligned}
a_{1+\alpha}(h)&=a_{1+\alpha}(0)+ha'_{1+\alpha}(0)+O(h^2)=a_0+h(1+\alpha)\dot a_0+O(h^2),\\
\psi_1(h)&=\psi_1(0)+h\psi_1'(0)+O(h^2)=\psi_0+h\dot\psi_0+O(h^2).
\end{aligned}$$

Hence, we get

$$a_{1+\alpha}(h)-a(t_0+(1+\alpha)h)=O(h^2),\quad \psi_1(h)-\psi(t_0+h)=O(h^2),$$

and

$$\begin{aligned}
z_1&=z_0+h\left((1-\gamma)a_\alpha+\gamma a_{1+\alpha}\right)\\
&=z_0+h\left((1-\gamma)(a_0+h\alpha\dot a_0+O(h^2))+\gamma(a_0+h(1+\alpha)\dot a_0+O(h^2))\right)\\
&=z_0+ha_0+h^2\underbrace{((1-\gamma)\alpha+\gamma(1+\alpha))}_{=\alpha+\gamma=\frac{1}{2}}\dot a_0+O(h^3)\\
&=z_0+ha_0+\frac{h^2}{2}\dot a_0+O(h^3),
\end{aligned}$$

leading to $z_1-z(t_0+h)=O(h^3)$. ▯

5. Perturbation analysis. Let

$$M := M(t,y), \quad F := f_\psi(t,y,z,\psi), \quad K := k_z(t,y,z).$$

We introduce the following projectors $Q = Q(t,y,z,\psi)$ and $P = P(t,y,z,\psi)$

$$Q := M^{-1}F(KM^{-1}F)^{-1}K, \quad P := I - Q.$$

They have the properties

$$\begin{array}{l} QM^{-1}F = M^{-1}F, \quad PM^{-1}F = O, \quad KQ = K, \quad KP = O, \\ Q^2 = Q, \quad QP = O, \quad P^2 = P, \quad PQ = O. \end{array}$$

The following lemma will be needed in the proof of the perturbation Theorem 5.2 below.

LEMMA 5.1. *Suppose $\gamma \neq 0$, $\alpha_m \neq 1$, and $\alpha_f \neq 1$. Using the notation just introduced above we have*

$$\begin{pmatrix} (1-\alpha_m)M & (1-\alpha_f)(-F) \\ \gamma K & O \end{pmatrix}^{-1} \begin{pmatrix} -\alpha_m M & -\alpha_f(-F) \\ -(1-\gamma)K & O \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\alpha_m}{(\alpha_m-1)}P + \frac{(\gamma-1)}{\gamma}Q & O \\ \nu(KM^{-1}F)^{-1}K & \frac{\alpha_f}{(\alpha_f-1)}I \end{pmatrix} \tag{5.1}$$

where $\nu := \frac{\alpha_m}{(1-\alpha_f)}\left(1 - \frac{(1-\gamma)(1-\alpha_m)}{\gamma\alpha_m}\right)$.

Proof. We can express

$$\begin{pmatrix} (1-\alpha_m)M & -(1-\alpha_f)F \\ \gamma K & O \end{pmatrix}$$

$$= \begin{pmatrix} (1-\alpha_m)I & O \\ O & \gamma I \end{pmatrix} \begin{pmatrix} M & -F \\ K & O \end{pmatrix} \begin{pmatrix} I & O \\ O & \frac{(1-\alpha_f)}{(1-\alpha_m)}I \end{pmatrix}.$$

Hence, we get

$$\begin{pmatrix} (1-\alpha_m)M & -(1-\alpha_f)F \\ \gamma K & O \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} I & O \\ O & \frac{(1-\alpha_m)}{(1-\alpha_f)}I \end{pmatrix} \begin{pmatrix} M & -F \\ K & O \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{(1-\alpha_m)}I & O \\ O & \frac{1}{\gamma}I \end{pmatrix}$$

where

$$\begin{pmatrix} M & -F \\ K & O \end{pmatrix}^{-1} = \begin{pmatrix} (I - M^{-1}F(KM^{-1}F)^{-1}K)M^{-1} & M^{-1}F(KM^{-1}F)^{-1} \\ -(KM^{-1}F)^{-1}KM^{-1} & (KM^{-1}F)^{-1} \end{pmatrix}.$$

The result (5.1) can then be obtained by direct calculation. ∎

THEOREM 5.2. *Consider $(\widetilde{y}_k, \widetilde{z}_k, \widetilde{\psi}_k)$ and $(\widehat{y}_k, \widehat{z}_k, \widehat{\psi}_k)$ at t_k and $\widetilde{a}_{k+\alpha}$, $\widehat{a}_{k+\alpha}$ at $t_{k+\alpha} = t_k + \alpha h$ satisfying for $h \to 0$*

$$k(t_k, \widetilde{y}_k, \widetilde{z}_k) = 0, \tag{5.2a}$$

$$k(t_k, \widehat{y}_k, \widehat{z}_k) = 0, \tag{5.2b}$$

$$\begin{array}{l} M(t_k, \widetilde{y}_k)\widetilde{a}_{k+\alpha} - f(t_k, \widetilde{y}_k, \widetilde{z}_k, \widetilde{\psi}_k) = o(1), \\ M(t_k, \widehat{y}_k)\widehat{a}_{k+\alpha} - f(t_k, \widehat{y}_k, \widehat{z}_k, \widehat{\psi}_k) = o(1), \\ k_t(t_k, \widetilde{y}_k, \widetilde{z}_k) + k_y(t_k, \widetilde{y}_k, \widetilde{z}_k)\widetilde{z}_k + k_z(t_k, \widetilde{y}_k, \widetilde{z}_k)\widetilde{a}_{k+\alpha} = o(1), \\ k_t(t_k, \widehat{y}_k, \widehat{z}_k) + k_y(t_k, \widehat{y}_k, \widehat{z}_k)\widehat{z}_k + k_z(t_k, \widehat{y}_k, \widehat{z}_k)\widehat{a}_{k+\alpha} = o(1). \end{array}$$

Suppose $\gamma \neq 0$, $\alpha_m \neq 1$, *and* $\alpha_f \neq 1$. *Let* $(\widetilde{y}_{k+1}, \widetilde{z}_{k+1}, \widetilde{\psi}_{k+1})$ $(\widehat{y}_{k+1}, \widehat{z}_{k+1}, \widehat{\psi}_{k+1})$ *and* $\widetilde{a}_{k+1+\alpha}$, $\widehat{a}_{k+1+\alpha}$, *be the corresponding generalized-*α *solutions (2.3) at* $t_{k+1} = t_k + h$ *and* $t_{k+1+\alpha} = t_k + (1+\alpha)h$ *respectively. Let* $\Delta y_k := \widehat{y}_k - \widetilde{y}_k$, $\Delta z_k := \widehat{z}_k - \widetilde{z}_k$, $\Delta a_{k+\alpha} := \widehat{a}_{k+\alpha} - \widetilde{a}_{k+\alpha}$, $\Delta\psi_k := \widehat{\psi}_k - \widetilde{\psi}_k$ *satisfying for* $h \to 0$

$$\Delta y_k = O(h), \quad \Delta z_k = O(h), \quad \Delta a_{k+\alpha} = O(h), \quad \Delta\psi_k = O(h).$$

Then we have

$$\begin{aligned}
\Delta y_{k+1} &= \Delta y_k + h\Delta z_k + \left(\frac{1}{2} + \frac{\beta}{(\alpha_m - 1)}\right) h^2 P_{k+\alpha}\Delta a_{k+\alpha} + \left(\frac{1}{2} - \frac{\beta}{\gamma}\right) h^2 Q_{k+\alpha}\Delta a_{k+\alpha} \\
&\quad + O\left(h^2\|\Delta y_k\| + h^2\|\Delta z_k\| + h^3\|\Delta a_{k+\alpha}\| + h^3\|\Delta\psi_k\| + h^2\|\Delta\psi_k\|^2\right), \\
\Delta z_{k+1} &= \Delta z_k + \left(1 + \frac{\gamma}{(\alpha_m - 1)}\right) hP_{k+\alpha}\Delta a_{k+\alpha} \\
&\quad + O\left(h\|\Delta y_k\| + h\|\Delta z_k\| + h^2\|\Delta a_{k+\alpha}\| + h^2\|\Delta\psi_k\| + h\|\Delta\psi_k\|^2\right), \\
h\Delta a_{k+1+\alpha} &= \frac{\alpha_m}{(\alpha_m - 1)} hP_{k+\alpha}\Delta a_{k+\alpha} + \frac{(\gamma - 1)}{\gamma} hQ_{k+\alpha}\Delta a_{k+\alpha} \\
&\quad + O\left(h\|\Delta y_k\| + h\|\Delta z_k\| + h^2\|\Delta a_{k+\alpha}\| + h^2\|\Delta\psi_k\| + h\|\Delta\psi_k\|^2\right), \\
h\Delta\psi_{k+1} &= -\frac{\alpha_f}{(1 - \alpha_f)} h\Delta\psi_k \\
&\quad + O\left(h\|\Delta y_k\| + \|\Delta y_k\|^2 + h\|\Delta z_k\| + \|\Delta z_k\|^2\right) \\
&\quad + O\left(h^2\|P_{k+\alpha}\Delta a_{k+\alpha}\| + h\|Q_{k+\alpha}\Delta a_{k+\alpha}\| + h^2\|\Delta\psi_k\| + h\|\Delta\psi_k\|^2\right)
\end{aligned}$$

where $P_{k+\alpha}$ *and* $Q_{k+\alpha}$ *are defined in the proof below in (5.7).*

Proof. By definition of the generalized-α method (2.3) we have

$$\Delta y_{k+1} = \Delta y_k + h\Delta z_k + \frac{h^2}{2}((1 - 2\beta)\Delta a_{k+\alpha} + 2\beta\Delta a_{k+1+\alpha}), \tag{5.3a}$$

$$\Delta z_{k+1} = \Delta z_k + h((1 - \gamma)\Delta a_{k+\alpha} + \gamma\Delta a_{k+1+\alpha}), \tag{5.3b}$$

and

$$\begin{aligned}
&(1 - \alpha_m)M_{k+1+\alpha}\Delta a_{k+1+\alpha} + \alpha_m M_{k+\alpha}\Delta a_{k+\alpha} + O(\|\Delta y_k\| + h\|\Delta z_k\|) \\
&= (1 - \alpha_f)\left(f_{y,k+1}\Delta y_{k+1} + f_{z,k+1}\Delta z_{k+1} + f_{\psi,k+1}\Delta\psi_{k+1}\right) \\
&\quad + \alpha_f\left(f_{y,k}\Delta y_k + f_{z,k}\Delta z_k + f_{\psi,k}\Delta\psi_k\right) \\
&\quad + O\left(\|\Delta y_k\|^2 + \|\Delta z_k\|^2 + \|\Delta\psi_k\|^2 + \|\Delta y_{k+1}\|^2 + \|\Delta z_{k+1}\|^2 + \|\Delta\psi_{k+1}\|^2\right)
\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}
&M_{k+1+\alpha} := M(t_{k+1+\alpha}, \widetilde{y}_k + h(1+\alpha)\widetilde{z}_k), \quad M_{k+\alpha} := M(t_{k+\alpha}, \widetilde{y}_k + h\alpha\widetilde{z}_k), \\
&f_{y,k} := f_y(t_k, \widetilde{y}_k, \widetilde{z}_k, \widetilde{\psi}_k), \quad f_{z,k} := f_z(t_k, \widetilde{y}_k, \widetilde{z}_k, \widetilde{\psi}_k), \quad f_{\psi,k} := f_\psi(t_k, \widetilde{y}_k, \widetilde{z}_k, \widetilde{\psi}_k),
\end{aligned}$$

and similarly for $f_{y,k+1}, f_{z,k+1}, f_{\psi,k+1}$. The relation (5.4) can be expressed

$$\begin{aligned}
&\begin{pmatrix} (1 - \alpha_m)M_{k+1+\alpha} & -(1 - \alpha_f)f_{\psi,k+1} \end{pmatrix} \begin{pmatrix} \Delta a_{k+1+\alpha} \\ \Delta\psi_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} -\alpha_m M_{k+\alpha} & \alpha_f f_{\psi,k} \end{pmatrix} \begin{pmatrix} \Delta a_{k+\alpha} \\ \Delta\psi_k \end{pmatrix} \\
&\quad + O\left(\|\Delta y_k\| + \|\Delta z_k\| + \|\Delta y_{k+1}\| + \|\Delta z_{k+1}\| + \|\Delta\psi_k\|^2 + \|\Delta\psi_{k+1}\|^2\right).
\end{aligned} \tag{5.5}$$

From

$$0 = k(t_{k+1}, \widetilde{y}_{k+1}, \widetilde{z}_{k+1}), \quad 0 = k(t_{k+1}, \widehat{y}_{k+1}, \widehat{z}_{k+1})$$

we have

$$0 = k_{y,k+1}\Delta y_{k+1} + k_{z,k+1}\Delta z_{k+1} + O\left(\|\Delta y_{k+1}\|^2 + \|\Delta z_{k+1}\|^2\right)$$

where

$$k_{y,k+1} := k_y(t_{k+1}, \widetilde{y}_{k+1}, \widetilde{z}_{k+1}), \quad k_{z,k+1} := k_z(t_{k+1}, \widetilde{y}_{k+1}, \widetilde{z}_{k+1}).$$

Introducing the expressions (5.3) for Δy_{k+1} and Δz_{k+1} we get

$$\begin{aligned} 0 =& k_{y,k+1}\left(\Delta y_k + h\Delta z_k + \frac{h^2}{2}((1-2\beta)\Delta a_{k+\alpha} + 2\beta\Delta a_{k+1+\alpha})\right) \\ &+ k_{z,k+1}\left(\Delta z_k + h((1-\gamma)\Delta a_{k+\alpha} + \gamma\Delta a_{k+1+\alpha})\right) + O\left(\|\Delta y_{k+1}\|^2 + \|\Delta z_{k+1}\|^2\right). \end{aligned}$$

From (5.2a) and (5.2b) we have

$$0 = k_{y,k}\Delta y_k + k_{z,k}\Delta z_k + O(\|\Delta y_k\|^2 + \|\Delta z_k\|^2),$$

and thus

$$\begin{aligned} k_{y,k+1}\Delta y_k + k_{z,k+1}\Delta z_k &= k_{y,k}\Delta y_k + k_{z,k}\Delta z_k + O(h\|\Delta y_k\| + h\|\Delta z_k\|) \\ &= O(h\|\Delta y_k\| + h\|\Delta z_k\| + \|\Delta y_k\|^2 + \|\Delta z_k\|^2). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &(\gamma k_{z,k+1} + h\beta k_{y,k+1})\, h\Delta a_{k+1+\alpha} \\ &= \left(-(1-\gamma)k_{z,k+1} - \frac{h}{2}(1-2\beta)k_{y,k+1}\right) h\Delta a_{k+\alpha} \\ &\quad + O\left(h\|\Delta y_k\| + h\|\Delta z_k\| + \|\Delta y_k\|^2 + \|\Delta z_k\|^2 + \|\Delta y_{k+1}\|^2 + \|\Delta z_{k+1}\|^2\right). \end{aligned} \tag{5.6}$$

In matrix form we obtain from (5.5) and (5.6)

$$\begin{aligned} &\begin{pmatrix} (1-\alpha_m)M_{k+1+\alpha} & -(1-\alpha_f)f_{\psi,k+1} \\ \gamma k_{z,k+1} + O(h) & O \end{pmatrix} \begin{pmatrix} h\Delta a_{k+1+\alpha} \\ h\Delta\psi_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} -\alpha_m M_{k+\alpha} & -\alpha_f(-f_{\psi,k}) \\ -(1-\gamma)k_{z,k+1} + O(h) & O \end{pmatrix} \begin{pmatrix} h\Delta a_{k+\alpha} \\ h\Delta\psi_k \end{pmatrix} \\ &\quad + \begin{pmatrix} O\left(h\|\Delta y_k\| + h\|\Delta z_k\| + h\|\Delta y_{k+1}\| + h\|\Delta z_{k+1}\| + h\|\Delta\psi_k\|^2 + h\|\Delta\psi_{k+1}\|^2\right) \\ O\left(h\|\Delta y_k\| + h\|\Delta z_k\| + \|\Delta y_k\|^2 + \|\Delta z_k\|^2 + \|\Delta y_{k+1}\|^2 + \|\Delta z_{k+1}\|^2\right) \end{pmatrix}. \end{aligned}$$

Let

$$\begin{aligned} &k_{z,k+\alpha} := k_z(t_{k+\alpha}, \widetilde{y}_{k+\alpha}, \widetilde{z}_{k+\alpha}), \quad f_{\psi,k+\alpha} := f_\psi(t_{k+\alpha}, \widetilde{y}_{k+\alpha}, \widetilde{z}_{k+\alpha}, \widetilde{\psi}_{k+\alpha}), \\ &Q_{k+\alpha} := Q(t_{k+\alpha}, \widetilde{y}_{k+\alpha}, \widetilde{z}_{k+\alpha}, \widetilde{\psi}_{k+\alpha}), \quad P_{k+\alpha} := P(t_{k+\alpha}, \widetilde{y}_{k+\alpha}, \widetilde{z}_{k+\alpha}, \widetilde{\psi}_{k+\alpha}) \end{aligned} \tag{5.7}$$

where $\widetilde{y}_{k+\alpha}, \widetilde{z}_{k+\alpha}, \widetilde{\psi}_{k+\alpha}$ can be taken for example as the exact solution at $t_{k+\alpha}$ passing through $\widetilde{y}_k, \widetilde{z}_k, \widetilde{\psi}_k$ at t_k. From

$$\begin{aligned} &k_{z,k} = k_{z,k+\alpha} + O(h), \quad k_{z,k+1} = k_{z,k+\alpha} + O(h), \\ &f_{\psi,k} = f_{\psi,k+\alpha} + O(h), \quad f_{\psi,k+1} = f_{\psi,k+\alpha} + O(h), \\ &M_{k+\alpha} = M(t_{k+\alpha}, \widetilde{y}_{k+\alpha}) + O(h), \quad M_{k+1+\alpha} = M(t_{k+\alpha}, \widetilde{y}_{k+\alpha}) + O(h), \end{aligned}$$

and (5.1) we obtain

$$\begin{aligned}
h\Delta a_{k+1+\alpha} =& \frac{\alpha_m}{(\alpha_m-1)} hP_{k+\alpha}\Delta a_{k+\alpha} + \frac{(\gamma-1)}{\gamma} hQ_{k+\alpha}\Delta a_{k+\alpha} \\
&+O\left(h\|\Delta y_k\| + h\|\Delta z_k\| + h^2\|\Delta a_{k+\alpha}\| + h^2\|\Delta\psi_k\| + h\|\Delta\psi_k\|^2\right) \\
&+O\left(h\|\Delta y_{k+1}\| + h\|\Delta z_{k+1}\| + h\|\Delta\psi_{k+1}\|^2\right), \\
h\Delta\psi_{k+1} =& \frac{\alpha_f}{(\alpha_f-1)} h\Delta\psi_k \\
&+O\left(h\|\Delta y_k\| + \|\Delta y_k\|^2 + h\|\Delta z_k\| + \|\Delta z_k\|^2\right) \\
&+O\left(h^2\|P_{k+\alpha}\Delta a_{k+\alpha}\| + h\|Q_{k+\alpha}\Delta a_{k+\alpha}\| + h^2\|\Delta\psi_k\| + h\|\Delta\psi_k\|^2\right) \\
&+O\left(h\|\Delta y_{k+1}\| + \|\Delta y_{k+1}\|^2 + h\|\Delta z_{k+1}\| + \|\Delta z_{k+1}\|^2 + h\|\Delta\psi_{k+1}\|^2\right).
\end{aligned}$$

These equations together with (5.3) lead to the desired result. ∎

6. Convergence and global error estimates. In the following theorem we prove second order of convergence of the generalized-α method (2.3) for systems with nonconstant mass matrix and nonholonomic constraints:

THEOREM 6.1. *Consider the system of index 2 DAEs (2.1) with consistent initial conditions (y_0, z_0, a_0, ψ_0) at t_0, see (2.2), and exact solution $(y(t), z(t), a(t), \psi(t))$. Suppose that $a_\alpha - a(t_0+\alpha h) = O(h)$ (e.g., $a_\alpha = a_0$), $\gamma \neq 0$, $\alpha_m \neq 1$, $\alpha_f \neq 1$, $\gamma = \frac{1}{2} - \alpha$,*

$$r_\gamma := \left|\frac{\gamma-1}{\gamma}\right| < 1, \quad r_{\alpha_m} := \left|\frac{\alpha_m}{\alpha_m-1}\right| < 1, \quad r_{\alpha_f} := \left|\frac{\alpha_f}{\alpha_f-1}\right| < 1,$$

and $r_\gamma \neq r_{\alpha_f}$. Then the generalized-$\alpha$ numerical approximation $(y_n, z_n, a_{n+\alpha}, \psi_n)$ (2.3) satisfies for $0 < |h| \leq h_{\max}$ and $|t_n - t_0| = n|h| \leq Const$, the following global error estimates

$$\begin{aligned}
&y_n - y(t_n) = O(h^2), \quad z_n - z(t_n) = O(h^2), \\
&a_{n+\alpha} - a(t_n+\alpha h) = O\left(h^2 + r_\gamma^n\delta_0\right), \quad \psi_n - \psi(t_n) = O\left(h^2 + (r_\gamma^n + r_{\alpha_f}^n)\delta_0\right)
\end{aligned}$$

where $\delta_0 := \|a_\alpha - a(t_0+\alpha h)\|$. Moreover, if $\delta_0 = O(h^2)$ then we have

$$a_{n+\alpha} - a(t_n+\alpha h) = O(h^2), \quad \psi_n - \psi(t_n) = O(h^2).$$

Proof. We consider two neighboring generalized-α approximations

$$(y_k^{k_1}, z_k^{k_1}, a_{k+\alpha}^{k_1}, \psi_k^{k_1})_{k=k_1}^n, \quad (y_k^{k_1-1}, z_k^{k_1-1}, a_{k+\alpha}^{k_1-1}, \psi_k^{k_1-1})_{k=k_1}^n$$

for $k_1 = 1, \dots, n$ and we denote their difference by

$$\begin{aligned}
&\Delta y_k := y_k^{k_1} - y_k^{k_1-1}, \quad \Delta z_k := z_k^{k_1} - z_k^{k_1-1}, \quad \Delta a_{k+\alpha} := a_{k+\alpha}^{k_1} - a_{k+\alpha}^{k_1-1}, \\
&\Delta\psi_k := \psi_k^{k_1} - \psi_k^{k_1-1}.
\end{aligned}$$

We assume that

$$\Delta y_k = O(h), \quad \Delta z_k = O(h), \quad \Delta a_{k+\alpha} = O(h), \quad \Delta\psi_k = O(h). \tag{6.1}$$

These assumptions can be justified by induction, see below. For $k = k_1$, Δy_{k_1}, Δz_{k_1}, $\Delta a_{k_1+\alpha}$, $\Delta\psi_{k_1}$ are just the local error (4.1) of the generalized-α method (2.3) with $(y_{k_1}^{k_1}, z_{k_1}^{k_1}, a_{k_1+\alpha}^{k_1}, \psi_{k_1}^{k_1})$ being the exact solution passing through $(y_{k_1-1}^{k_1-1}, z_{k_1-1}^{k_1-1}, \psi_{k_1-1}^{k_1-1})$ and $(y_{k_1}^{k_1-1}, z_{k_1}^{k_1-1}, a_{k_1+\alpha}^{k_1-1}, \psi_{k_1}^{k_1-1})$ being the generalized-α numerical approximation from the same point. The generalized-α approximations satisfy the nonholonomic constraints (2.1b). Using the decomposition

$$\Delta a_{k+1+\alpha} = P_{k+1+\alpha}\Delta a_{k+1+\alpha} + Q_{k+1+\alpha}\Delta a_{k+1+\alpha}.$$

we get by application of Theorem 5.2 for $k = k_1, \dots, n-1$

$$\begin{aligned}
\Delta y_{k+1} &= \Delta y_k + h\Delta z_k + \left(\frac{1}{2} + \frac{\beta}{(\alpha_m - 1)}\right) h^2 P_{k+\alpha}\Delta a_{k+\alpha} \\
&\qquad + \left(\frac{1}{2} - \frac{\beta}{\gamma}\right) h^2 Q_{k+\alpha}\Delta a_{k+\alpha} \\
&\quad + O\left(h^2\|\Delta y_k\| + h^2\|\Delta z_k\| + h^3\|\Delta a_{k+\alpha}\| + h^3\|\Delta\psi_k\|\right), \\
\Delta z_{k+1} &= \Delta z_k + \left(1 + \frac{\gamma}{(\alpha_m - 1)}\right) hP_{k+\alpha}\Delta a_{k+\alpha} \\
&\quad + O\left(h\|\Delta y_k\| + h\|\Delta z_k\| + h^2\|\Delta a_{k+\alpha}\| + h^2\|\Delta\psi_k\|\right), \\
hP_{k+1+\alpha}\Delta a_{k+1+\alpha} &= \frac{\alpha_m}{(\alpha_m - 1)} hP_{k+\alpha}\Delta a_{k+\alpha} \\
&\quad + O\left(h\|\Delta y_k\| + h\|\Delta z_k\| + h^2\|\Delta a_{k+\alpha}\| + h^2\|\Delta\psi_k\|\right), \\
hQ_{k+1+\alpha}\Delta a_{k+1+\alpha} &= \frac{(\gamma - 1)}{\gamma} hQ_{k+\alpha}\Delta a_{k+\alpha} \\
&\quad + O\left(h\|\Delta y_k\| + h\|\Delta z_k\| + h^2\|\Delta a_{k+\alpha}\| + h^2\|\Delta\psi_k\|\right), \\
h\Delta\psi_{k+1} &= \frac{\alpha_f}{(\alpha_f - 1)} h\Delta\psi_k \\
&\quad + O\left(h\|\Delta y_k\| + h\|\Delta z_k\|\right) \\
&\quad + O\left(h^2\|P_{k+\alpha}\Delta a_{k+\alpha}\| + h\|Q_{k+\alpha}\Delta a_{k+\alpha}\| + h^2\|\Delta\psi_k\|\right).
\end{aligned}$$

Taking a norm of these expressions leads to the estimates

$$\begin{pmatrix} \|\Delta y_{k+1}\| \\ \|\Delta z_{k+1}\| \\ h\|P_{k+1+\alpha}\Delta a_{k+1+\alpha}\| \\ h\|Q_{k+1+\alpha}\Delta a_{k+1+\alpha}\| \\ h\|\Delta\psi_{k+1}\| \end{pmatrix} \le M \begin{pmatrix} \|\Delta y_k\| \\ \|\Delta z_k\| \\ h\|P_{k+\alpha}\Delta a_{k+\alpha}\| \\ h\|Q_{k+\alpha}\Delta a_{k+\alpha}\| \\ h\|\Delta\psi_k\| \end{pmatrix}$$

with matrix

$$M := \begin{pmatrix} 1 + O(h^2) & h + O(h^2) & O(h) & O(h) & O(h^2) \\ O(h) & 1 + O(h) & c + O(h) & O(h) & O(h) \\ O(h) & O(h) & r_{\alpha_m} + O(h) & O(h) & O(h) \\ O(h) & O(h) & O(h) & r_\gamma + O(h) & O(h) \\ O(h) & O(h) & O(h) & d & r_{\alpha_f} + O(h) \end{pmatrix}.$$

where $c := \left|1 + \frac{\gamma}{(\alpha_m - 1)}\right|$ and $d = O(1)$ can be fixed as a constant independently of k_1 and k. Defining the matrix

$$T := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{c}{(r_{\alpha_m}-1)} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{d}{(r_\gamma - r_{\alpha_f})} & 1 \end{pmatrix},$$

we can transform the matrix M by a similarity transformation to the form

$$\begin{aligned} N := & T^{-1}MT \\ = & \begin{pmatrix} 1+O(h^2) & h+O(h^2) & O(h) & O(h) & O(h^2) \\ O(h) & 1+O(h) & O(h) & O(h) & O(h) \\ O(h) & O(h) & r_{\alpha_m}+O(h) & O(h) & O(h) \\ O(h) & O(h) & O(h) & r_\gamma+O(h) & O(h) \\ O(h) & O(h) & O(h) & O(h) & r_{\alpha_f}+O(h) \end{pmatrix}. \end{aligned}$$

For this matrix N there is a first linear invariant subspace $V_{12} \subset \mathbb{R}^4$ associated to the two eigenvalues $\mu_1 = 1 + O(h)$ and $\mu_2 = 1 + O(h)$. This subspace V_{12} is of the form $V_{12} = \mathrm{span}(e_1 + O(h), e_2 + O(h))$ where $e_1 := (1,0,0,0,0)^T \in \mathbb{R}^5$ and $e_2 := (0,1,0,0,0)^T \in \mathbb{R}^5$. There is a second linear invariant subspace $V_3 \subset \mathbb{R}^5$ associated to the eigenvalue $\mu_3 = r_{\alpha_m} + O(h)$. This subspace V_{34} is of the form $V_3 = \mathrm{span}(e_3 + O(h))$ where $e_3 := (0,0,1,0,0)^T \in \mathbb{R}^5$. There is a third linear invariant subspace $V_4 \subset \mathbb{R}^5$ associated to the eigenvalue $\mu_4 = r_\gamma + O(h)$. This subspace V_4 is of the form $V_4 = \mathrm{span}(e_4 + O(h))$ where $e_4 := (0,0,0,1,0)^T \in \mathbb{R}^5$. There is finally a fourth linear invariant subspace $V_5 \subset \mathbb{R}^5$ associated to the eigenvalue $\mu_5 = r_{\alpha_f} + O(h)$. This subspace V_5 is of the form $V_5 = \mathrm{span}(e_5 + O(h))$ where $e_5 := (0,0,0,0,1)^T \in \mathbb{R}^5$. Therefore, there is a transformation $V = I + O(h)$ with inverse $V^{-1} = I + O(h)$ such that $NV = VB$ with B block-diagonal, i.e.,

$$B := V^{-1}NV = \begin{pmatrix} 1+O(h) & O(h) & 0 & 0 & 0 \\ O(h) & 1+O(h) & 0 & 0 & 0 \\ 0 & 0 & r_{\alpha_m}+O(h) & 0 & 0 \\ 0 & 0 & 0 & r_\gamma+O(h) & 0 \\ 0 & 0 & 0 & 0 & r_{\alpha_f}+O(h) \end{pmatrix}.$$

For $2 \le m \le n$, from $mh \le nh \le Const$ we obtain

$$\begin{pmatrix} \|\Delta y_n\| \\ \|\Delta z_n\| \\ h\|P_{n+\alpha}\Delta a_{n+\alpha}\| \\ h\|Q_{n+\alpha}\Delta a_{n+\alpha}\| \\ h\|\Delta\psi_n\| \end{pmatrix} \le M^m \begin{pmatrix} \|\Delta y_{n-m}\| \\ \|\Delta z_{n-m}\| \\ h\|P_{n-m+\alpha}\Delta a_{n-m+\alpha}\| \\ h\|Q_{n-m+\alpha}\Delta a_{n-m+\alpha}\| \\ h\|\Delta\psi_{n-m}\| \end{pmatrix}.$$

where

$$\begin{aligned} M^m = & TVB^mV^{-1}T^{-1} \\ = & \begin{pmatrix} O(1) & O(1) & O(1) & O(h) & O(h) \\ O(1) & O(1) & O(1) & O(h) & O(h) \\ O(h) & O(h) & O(h) & O(hr_\gamma^m + hr_{\alpha_f}^m + h^2) & O(hr_{\alpha_f}^m + h^2) \\ O(h) & O(h) & O(h) & O(r_\gamma^m + hr_{\alpha_f}^m + h^2) & O(hr_{\alpha_f}^m + h^2) \\ O(h) & O(h) & O(h) & O(r_\gamma^m + r_{\alpha_f}^m + h^2) & O(r_{\alpha_f}^m + h^2) \end{pmatrix}. \end{aligned}$$

Hence, we get

(6.2a) $\|\Delta y_n\| \leq C\left(\|\Delta y_{n-m}\| + \|\Delta z_{n-m}\| + h\|\Delta a_{n-m+\alpha}\| + h^2\|\Delta\psi_{n-m}\|\right),$

(6.2b) $\|\Delta z_n\| \leq C\left(\|\Delta y_{n-m}\| + \|\Delta z_{n-m}\| + h\|\Delta a_{n-m+\alpha}\| + h^2\|\Delta\psi_{n-m}\|\right),$

(6.2c) $\|\Delta a_n\| \leq C\big(\|\Delta y_{n-m}\| + \|\Delta z_{n-m}\| + (r_\gamma^m + h)\|\Delta a_{n-m+\alpha}\| + (r_{\alpha_f}^m + h)h\|\Delta\psi_{n-m}\|\big),$

(6.2d) $\|\Delta\psi_n\| \leq C\big(\|\Delta y_{n-m}\| + \|\Delta z_{n-m}\| + (r_\gamma^m + r_{\alpha_f}^m + h)\|\Delta a_{n-m+\alpha}\| + (r_{\alpha_f}^m + h)\|\Delta\psi_{n-m}\|\big).$

First, we consider the generalized-α solution using the exact value $a(t_0+\alpha h)$. We denote it by $(\overline{y}_k, \overline{z}_k, \overline{a}_{k+\alpha}, \overline{\psi}_k)_{k=0}^n$. For $k=0$ we have $\overline{y}_0 = y_0$, $\overline{z}_0 = z_0$, $\overline{a}_\alpha = a(t_0+\alpha h)$, and $\overline{\psi}_0 = \psi_0$. Taking $m := k_1$ in (6.2) leads to

$$\|\Delta y_n\| \leq c_y h^3, \qquad \|\Delta z_n\| \leq c_z h^3,$$
$$\|\Delta a_n\| \leq c_a(h^3 + r_\gamma^{k_1} h^2), \qquad \|\Delta\psi_n\| \leq c_\psi(h^3 + (r_\gamma^{k_1} + r_{\alpha_f}^{k_1})h^2)$$

which is also valid for $k_1 = n$ $(m=0)$ and $k_1 = n-1$ $(m=1)$. The assumptions (6.1) are thus justified by induction on k. Summing up these estimates we obtain

$$\|y(t_n) - \overline{y}_n\| \leq \sum_{k_1=1}^{n} \|y_n^{k_1} - y_n^{k_1-1}\| \leq c_y n h^3 \leq C_y h^2,$$

$$\|z(t_n) - \overline{z}_n\| \leq \sum_{k_1=1}^{n} \|z_n^{k_1} - z_n^{k_1-1}\| \leq c_z n h^3 \leq C_z h^2,$$

$$\|a(t_n+\alpha h) - \overline{a}_{n+\alpha}\| \leq \sum_{k_1=1}^{n} \|a_{n+\alpha}^{k_1} - a_{n+\alpha}^{k_1-1}\| \leq c_a\left(nh^3 + h^2\sum_{k_1=1}^{n} r_\gamma^{k_1}\right)$$
$$\leq c_a\left(nh^3 + h^2\frac{r_\gamma}{1-r_\gamma}\right) \leq C_a h^2,$$

$$\|\psi(t_n) - \overline{\psi}_n\| \leq \sum_{k_1=1}^{n} \|\psi_n^{k_1} - \psi_n^{k_1-1}\| \leq c_\psi\left(nh^3 + h^2\sum_{k_1=1}^{n}(r_\gamma^{k_1} + r_{\alpha_f}^{k_1})\right)$$
$$\leq c_\psi\left(nh^3 + h^2\left(\frac{r_\gamma}{1-r_\gamma} + \frac{r_{\alpha_f}}{1-r_{\alpha_f}}\right)\right) \leq C_\psi h^2.$$

Now, suppose that a_α satisfies $a_\alpha = \overline{a}_\alpha + O(h)$ where $\overline{a}_\alpha := a(t_0+\alpha h)$. We denote the corresponding generalized-α solution using this approximate value of a_α by $(y_k, z_k, a_{\alpha+k}, \psi_k)_{k=0}^n$. We want to estimate $\|y_n - \overline{y}_n\|$, $\|z_n - \overline{z}_n\|$, $\|a_{\alpha+n} - \overline{a}_{\alpha+n}\|$, and $\|\psi_n - \overline{\psi}_n\|$. Using (6.2) for $m=n$, since $\overline{y}_0 = y_0$, $\overline{z}_0 = z_0$, and $\overline{\psi}_0 = \psi_0$ we simply obtain

$$\|y_n - \overline{y}_n\| \leq Ch\|a_\alpha - \overline{a}_\alpha\|,$$
$$\|z_n - \overline{z}_n\| \leq Ch\|a_\alpha - \overline{a}_\alpha\|,$$
$$\|a_{n+\alpha} - \overline{a}_{n+\alpha}\| \leq C(r_\gamma^n + h)\|a_\alpha - \overline{a}_\alpha\|,$$
$$\|\psi_n - \overline{\psi}_n\| \leq C(r_\gamma^n + r_{\alpha_f}^n + h)\|a_\alpha - \overline{a}_\alpha\|.$$

The assumptions (6.1) are still justified by induction on k. By combining the above estimates

$$\|y_n - y(t_n)\| = \|y_n - \overline{y}_n\| + \|\overline{y}_n - y(t_n)\|,$$

$$\|z_n - z(t_n)\| = \|z_n - \overline{z}_n\| + \|\overline{z}_n - z(t_n)\|,$$
$$\|a_{n+\alpha} - a(t_n+\alpha h)\| = \|a_{n+\alpha} - \overline{a}_{n+\alpha}\| + \|\overline{a}_{n+\alpha} - a(t_n+\alpha h)\|,$$
$$\|\psi_n - \psi(t_n)\| = \|\psi_n - \overline{\psi}_n\| + \|\overline{\psi}_n - \psi(t_n)\|,$$

we finally obtain the desired result. ∎

7. Numerical Experiments. To illustrate Theorem 6.1 numerically we consider the following mathematical test problem

$$\begin{pmatrix} \dot y_1 \\ \dot y_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{7.1a}$$

$$\begin{pmatrix} y_1 & y_2 - e^{-2t} \\ \sin(y_1 - e^t) & y_1 y_2 \end{pmatrix}\begin{pmatrix} \dot z_1 \\ \dot z_2 \end{pmatrix} = \begin{pmatrix} e^t(y_1 z_2 + 2 y_2 z_1) + e^{2t} y_1 \psi_1 \\ e^{-t}(0.5 y_2 z_2 - 2 y_1 z_1 y_2 z_2 + y_2 \psi_1^2) \end{pmatrix}, \tag{7.1b}$$

$$k(t,y,z) = z_1^2 z_2 + 6 y_1 y_2 z_1 - 4 = 0. \tag{7.1c}$$

Observe that this problem is nonlinear in the algebraic variable ψ_1. The following consistent initial conditions at $t_0 = 0$ have been used

$$y_0 = \begin{pmatrix} 1, & 1 \end{pmatrix}^T, \qquad z_0 = \begin{pmatrix} 1, & -2 \end{pmatrix}^T.$$

The exact solution is given explicitly as follows

$$y(t) = \begin{pmatrix} e^t, & e^{-2t} \end{pmatrix}^T, \qquad z(t) = \begin{pmatrix} e^t, & -2e^{-2t} \end{pmatrix}^T, \qquad \psi(t) = \begin{pmatrix} e^{-t} \end{pmatrix}^T.$$

We have applied the generalized-α method (2.3) with damping parameter $\rho_\infty = 0.2$ and constant stepsizes h for various values of h. We observe global convergence of order 2 at $t_n = 1$ in Fig. 7.1.

For the second numerical experiment we consider the equations of a rolling disk, see, e.g., [1]. These equations can be expressed for $y, z \in \mathbb{R}^5, \psi \in \mathbb{R}^2$ in the form (2.1) with

$$M(t,y) = \begin{pmatrix} m & 0 & -mrc_3s_4 & -mrs_3c_4 & 0 \\ 0 & m & mrc_3c_4 & -mrs_3s_4 & 0 \\ -mrc_3s_4 & mrc_3c_4 & mr^2 + I_1 & 0 & 0 \\ -mrs_3c_4 & -mrs_3s_4 & 0 & mr^2s_3^2 + I_1c_3^2 + I_2s_3^2 & I_2s_3 \\ 0 & 0 & 0 & I_2s_3 & I_2 \end{pmatrix},$$

$$\begin{aligned}
f_1(t,y,z,\psi) &= +mr\big(-z_3s_3s_4 + z_4c_3c_4\big)z_3 + mr\big(z_3c_3c_4 - z_4s_3s_4\big)z_4 - \psi_1, \\
f_2(t,y,z,\psi) &= +mr\big(z_3s_3c_4 + z_4c_3s_4\big)z_3 + mr\big(z_3c_3s_4 + z_4s_3c_4\big)z_4 - \psi_2, \\
f_3(t,y,z,\psi) &= mr^2z_4^2s_3c_3 - mr\big(-z_3s_3(z_1s_4 - z_2c_4) + z_4c_3(z_1c_4 + z_2s_4)\big) \\
&\quad -I_1z_4^2c_3s_3 + I_2\big(z_5 + z_4s_3\big)z_4c_3 + mrgs_3 \\
&\quad -mrs_3\big(z_1s_4 - z_2c_4\big)z_3 + mrc_3\big(z_1c_4 + z_2s_4\big)z_4, \qquad (7.2) \\
f_4(t,y,z,\psi) &= -mr\big(z_3c_3(z_1c_4 + z_2s_4) + z_4s_3(-z_1s_4 + z_2c_4)\big) \\
&\quad -\big(2mr^2z_4s_3c_3 - mrc_3(z_1c_4 + z_2s_4) - 2I_1z_4s_3c_3 + 2I_2z_4s_3c_3\big)z_3 \\
&\quad +mrs_3(-z_1s_4 + z_2c_4)z_4, \\
f_5(t,y,z,\psi) &= -I_2z_4c_3z_3 + r(c_4\psi_1 + s_4\psi_2), \\
k_1(t,y,z,\psi) &= z_1 - rc_4z_5 = 0, \\
k_2(t,y,z,\psi) &= z_2 - rs_4z_5 = 0,
\end{aligned}$$

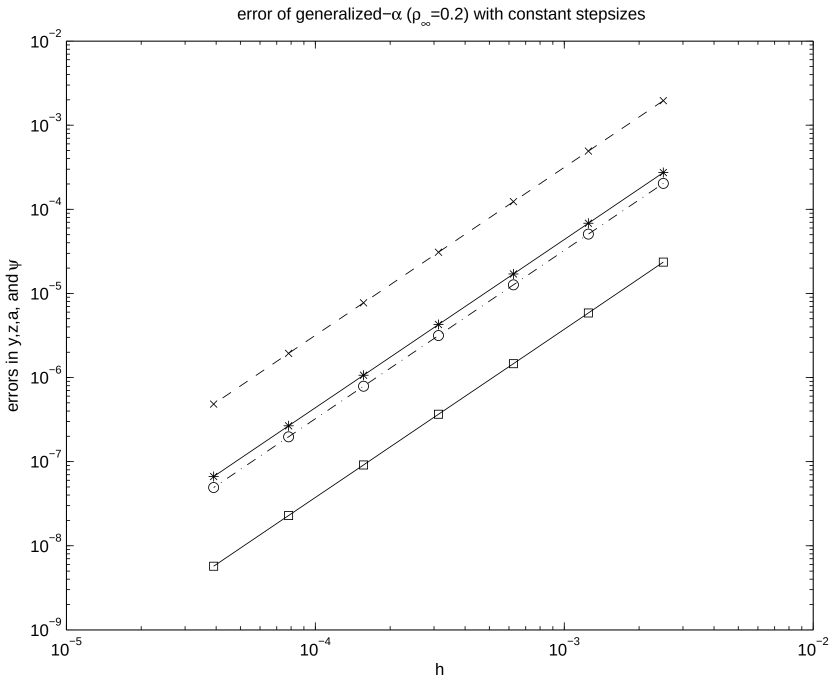

FIG. 7.1. *Global errors* $\|y_n - y(t_n)\|_2$ ($\square$), $\|z_n - z(t_n)\|_2$ ($\circ$), $\|a_{n+\alpha} - a(t_n + \alpha h)\|_2$ ($\times$), $\|\psi_n - \psi(t_n)\|_2$ ($*$) *of the generalized-*α *method (2.3) with* $\rho_\infty = 0.2$ *at* $t_n = 1$ *for the test problem (7.1) with various constant step sizes* h.

where we have used the notation $c_3 := \cos(y_3)$, $s_3 := \sin(y_3)$, $c_4 := \cos(y_4)$, and $s_4 := \sin(y_4)$. These equations correspond to a Lagrangian system with nonholonomic constraints

$$\begin{aligned} \frac{d}{dt}y &= z, \\ \frac{d}{dt}\nabla_z L(t,y,z) &= \nabla_y L(t,y,z) - k_z(t,y,z)^T\psi, \\ 0 &= k(t,y,z), \end{aligned}$$

with Lagrangian $L(t,y,z) = T(t,y,z) - U(t,y,z)$ where $T(t,y,z)$ and $U(t,y,z)$ are given here by

$$\begin{aligned} T(t,y,z) =& \frac{1}{2}m(z_1^2 + z_2^2 + r^2 z_3^2 + r^2 z_4^2 \sin^2(y_3)) - mr(z_3\cos(y_3)(z_1\sin(y_4) - z_2\cos(y_4)) \\ & + z_4\sin(y_3)(z_1\cos(y_4) + z_2\sin(y_4))) \\ & + \frac{1}{2}I_1(z_3^2 + z_4^2\cos(y_3)^2) + \frac{1}{2}I_2(z_5 + z_4\sin(y_3))^2, \\ U(t,y,z) =& mgr\cos(y_3). \end{aligned}$$

We have considered the parameters $m = 2$, $r = 1$, $I_1 = 2$, $I_2 = 2$, $g = 10$, and the following consistent initial conditions at $t_0 = 0$

$$y_0 = \begin{pmatrix} 0.1, & 0, & 0.3, & 0, & 1 \end{pmatrix}^T, \qquad z_0 = \begin{pmatrix} 0.1, & 0, & 0.02, & -0.02, & 0.1 \end{pmatrix}^T.$$

We have applied the generalized-α method (2.3) with damping parameter $\rho_\infty = 0.2$ and constant stepsizes h for various values of h. We observe again global convergence of order 2 at $t_n = 10$ in Fig. 7.2.

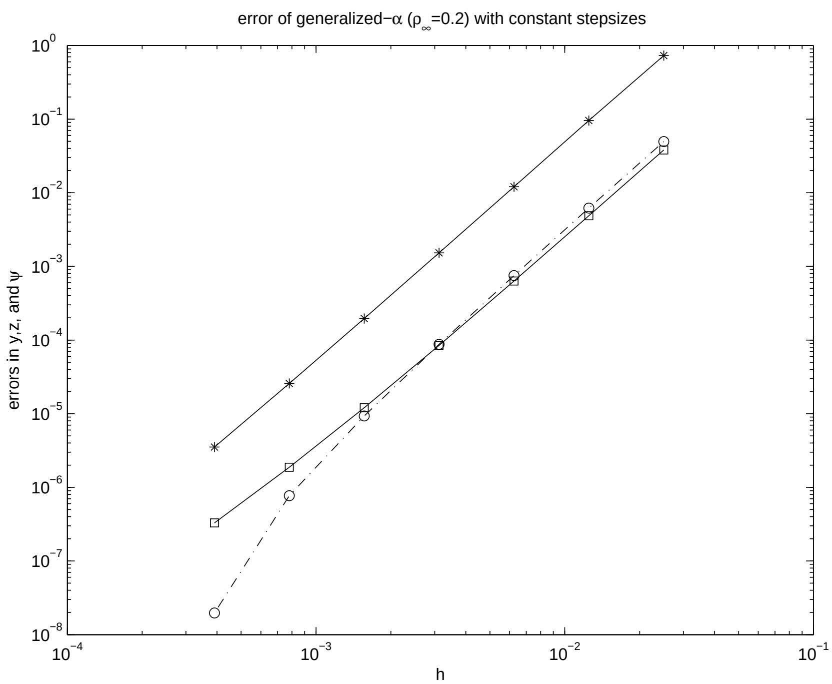

FIG. 7.2. *Global errors* $\|y_n - y(t_n)\|_2$ (□), $\|z_n - z(t_n)\|_2$ (○), $\|\psi_n - \psi(t_n)\|_2$ ($*$) *of the generalized-*α *method (2.3) with* $\rho_\infty = 0.2$ *at* $t_n = 10$ *for the rolling disk problem (7.2) with various constant step sizes* h.

8. Conclusions. The generalized-α method of Chung and Hulbert [2] is extended to systems having nonconstant mass matrix and nonholonomic constraints. The extension of the generalized-α method analyzed in this paper is shown to be second order which has been illustrated by two numerical experiments.

REFERENCES

[1] A. M. BLOCH, *Nonholonomic mechanics and control*, vol. 24 of Interdisciplinary Applied Mathematics, Springer, 2003.

[2] J. CHUNG AND G. M. HULBERT, *A time integration algorithm for structural dynamics with improved numerical dissipation: the generalized-α method*, J. Appl. Mech., 60 (1993), pp. 371–375.

[3] H. M. HILBER, T. J. R. HUGHES, AND R. L. TAYLOR, *Improved numerical dissipation for time integration algorithms in structural dynamics*, Earthquake Engng Struct. Dyn., 5 (1977), pp. 283–292.

[4] T. J. R. HUGHES, *Finite element method - Linear static and dynamic finite element analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1987.

[5] L. O. JAY AND D. NEGRUT, *A second order extension of the generalized-α method for constrained systems in mechanics*, in Multibody Dynamics, Computational Methods and Applications, C. L. Bottasso, ed., vol. 12 of Computational Methods in Applied Sciences, Springer, E. Onate series ed., 2008, pp. 143–158.

[6] A. A. Shabana, *Dynamics of multibody systems*, Cambridge University Press, third edition, 2005.