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A SECOND ORDER EXTENSION OF THE GENERALIZED-α METHOD FOR CONSTRAINED SYSTEMS IN MECHANICS

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Abstract. *We present a new second order extension of the generalized-α method for systems in mechanics with a nonconstant mass matrix, holonomic constraints, and nonholonomic constraints. A new variable stepsizes formula preserving the second order of the method is also proposed.*

1 INTRODUCTION

We consider second order systems of differential equations of the form $My'' = f(t, y, y')$. In mechanics $M \in \mathbb{R}^{n\times n}$ is a constant mass matrix, $y \in \mathbb{R}^n$ is a vector of generalized coordinates, $y' \in \mathbb{R}^n$ is a vector of generalized velocities, and $y'' \in \mathbb{R}^n$ is a vector of generalized accelerations. Introducing the new variables $z := y' \in \mathbb{R}^n$ and $a := z' = y'' \in \mathbb{R}^n$, these equations are equivalent to the semi-explicit system of differential-algebraic equations (DAEs)

$$y' = z, \quad z' = a, \quad 0 = Ma - f(t, y, z). \tag{1}$$

Assuming the mass matrix M to be invertible, the system of DAEs given by Eq. (1) is of index 1 since one can obtain explicitly the relation $a = M^{-1}f(t, y, z)$. The generalized-α method of Chung and Hulbert (see Ref. [2]) for $My'' = f(t, y, y')$ or equivalently for Eq. (1) is a non-standard implicit one-step method. One step of the method $(t_0, y_0, z_0, a_\alpha) \mapsto (t_1 = t_0 + h, y_1, z_1, a_{1+\alpha})$ with stepsize h can be expressed as follows

$$\begin{aligned} y_1 &= y_0 + hz_0 + \frac{h^2}{2}\left((1-2\beta)a_\alpha + 2\beta a_{1+\alpha}\right), & (2)\\ z_1 &= z_0 + h\left((1-\gamma)a_\alpha + \gamma a_{1+\alpha}\right), & (3)\\ (1-\alpha_m)Ma_{1+\alpha} + \alpha_m M a_\alpha &= (1-\alpha_f)f(t_1, y_1, z_1) + \alpha_f f(t_0, y_0, z_0), & (4)\end{aligned}$$

see section 2 below for a justification of the notation $a_\alpha, a_{1+\alpha}$. The generalized-α method has coefficients $\beta, \gamma, \alpha_m \neq 1, \alpha_f$. For certain speficic choices of these coefficients we obtain well-known methods:

- Newmark's family: $\alpha_m = 0$, $\alpha_f = 0$;
  - Trapezoidal rule: $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$;
  - Störmer's rule: $\beta = 0$, $\gamma = \frac{1}{2}$;
- The Hilber-Hughes-Taylor α (HHT-α) method (see Refs. [3, 4]):

$$\alpha_m = 0, \quad \alpha = -\alpha_f \in \left[-\frac{1}{3}, 0\right], \quad \beta = \frac{(1-\alpha)^2}{4}, \quad \gamma = \frac{1}{2} - \alpha.$$

The coefficients $\alpha_m, \alpha_f, \beta, \gamma$ of the generalized-α method in Eq. (2-4) are usually chosen according to

$$\alpha_m = \frac{2\rho_\infty - 1}{1+\rho_\infty}, \qquad \alpha_f = \frac{\rho_\infty}{1+\rho_\infty}, \qquad \beta = \frac{(1-\alpha)^2}{4}, \qquad \gamma = \frac{1}{2} - \alpha,$$

where $\alpha := \alpha_m - \alpha_f$ and $\rho_\infty \in [0, 1]$ is a parameter controlling numerical dissipation ($\rho_\infty = 0$ for maximal dissipation, see Ref. [2]).

In this paper we present extensions of the generalized-α method of Eqs. (2-4) for

- nonconstant mass matrix $M(t, y)$;
- holonomic constraints $g(t, y) = 0$;
- nonholonomic constraints $k(t, y, y') = 0$;
- variable stepsizes h_n.

2 ABOUT THE NOTATION $a_\alpha, a_{1+\alpha}$

We use the notation a_α and $a_{1+\alpha}$ instead of a_0 and a_1 to emphasize the fact that these quantities should not be considered as approximations to the acceleration vector $a(t)$ at t_0 and t_1 respectively, but at $t_\alpha := t_0 + \alpha h$ and $t_{1+\alpha} := t_1 + \alpha h = t_0 + (1+\alpha)h$ respectively where $\alpha := \alpha_m - \alpha_f$. The reason is that for a solution $(y(t), z(t), a(t))$ and values (y_0, z_0) satisfying $y_0 - y(t_0) = O(h^2)$, $z_0 - z(t_0) = O(h^2)$, we have

$$a_{1+\alpha} - a(t_{1+\alpha}) = O(h^2) \quad \text{when} \quad a_\alpha - a(t_\alpha) = O(h^2), \tag{5}$$

whereas we only have $a_{1+\alpha} - a(t_1) = O(h)$ when $a_\alpha - a(t_0) = O(h^2)$ and $\alpha \neq 0$. This can be seen as follows. We rewrite Eq. (4) as

$$(1-\alpha_m)a_{1+\alpha} + \alpha_m a_\alpha = (1-\alpha_f)M^{-1}f(t_1, y_1, z_1) + \alpha_f M^{-1}f(t_0, y_0, z_0). \tag{6}$$

Since $a(t) = M^{-1}f(t, y(t), z(t))$, $y_1 - y(t_1) = O(h^2)$, and $z_1 - z(t_1) = O(h^2)$ we have

$$M^{-1}f(t_1, y_1, z_1) = a(t_0) + ha'(t_0) + O(h^2), \quad M^{-1}f(t_0, y_0, z_0) = a(t_0) + O(h^2).$$

Hence, for the right-hand side of Eq. (6) we obtain

$$(1-\alpha_f)M^{-1}f(t_1, y_1, z_1) + \alpha_f M^{-1}f(t_0, y_0, z_0) = a(t_0) + h(1-\alpha_f)a'(t_0) + O(h^2). \tag{7}$$

Since

$$a(t_{1+\alpha}) = a(t_0) + h(1+\alpha)a'(t_0) + O(h^2), \quad a(t_\alpha) = a(t_0) + h\alpha a'(t_0) + O(h^2),$$

we have

$$(1-\alpha_m)a(t_{1+\alpha}) + \alpha_m a(t_\alpha) = a(t_0) + h(1-\alpha_m+\alpha)a'(t_0) + O(h^2). \tag{8}$$

Thus, from Eqs. (6-7-8), we obtain

$$(1-\alpha_m)(a_{1+\alpha} - a(t_{1+\alpha})) + \alpha_m(a_\alpha - a(t_\alpha)) = h(-\alpha_f + \alpha_m - \alpha)a'(t_0) + O(h^2). \tag{9}$$

Hence, Eq. (5) is satisfied only for $\alpha = \alpha_m - \alpha_f$.

2.1 Defining a_α for the first step

The definition of a_α for the first step remains. For $\alpha_m = 0$, for example for the HHT-α method, we see from Eq. (9) that taking $a_\alpha = a_0$ where $Ma_0 = f(t_0, y_0, z_0)$ still leads to the estimate $a_{1+\alpha} - a(t_{1+\alpha}) = O(h^2)$. When $\alpha_m \neq 0$ it is better to define a_α such that $a_\alpha - a(t_\alpha) = O(h^2)$, for example implicitly by

$$Ma_\alpha = (1-\alpha)f(t_0, y_0, z_0) + \alpha f(t_1, y_1, z_1) \tag{10}$$

as proposed by Lunk and Simeon in Ref. [7]. Nevertheless, taking $a_\alpha = a_0$ does not affect the order of global convergence of the y and z components, see Theorem 1 below.

3 NONCONSTANT MASS MATRIX $M(t,y)$

We consider $M(t,y)y'' = f(t,y,y')$ where $M(t,y)$ is a nonconstant mass matrix assumed to be invertible. These equations are equivalent to the semi-explicit system of index 1 DAEs

$$y' = z, \quad z' = a, \quad 0 = M(t,y)a - f(t,y,z).$$

A natural extension of the generalized-α method of Eqs. (2-4) is to replace Eq. (4) with

$$(1-\alpha_m)M_{1+\alpha}a_{1+\alpha} + \alpha_m M_\alpha a_\alpha = (1-\alpha_f)f(t_1,y_1,z_1) + \alpha_f f(t_0,y_0,z_0)$$

where

$$M_{1+\alpha} \approx M(t_{1+\alpha}, y(t_{1+\alpha})), \quad M_\alpha \approx M(t_\alpha, y(t_\alpha)).$$

For example we can take explicitly

$$M_{1+\alpha} := M(t_{1+\alpha}, y_0 + h(1+\alpha)z_0), \quad M_\alpha := M_{(1+\alpha)-1} \text{ or } M(t_\alpha, y_0 + h\alpha z_0)$$

where $M_{(1+\alpha)-1}$ denotes the matrix $M_{1+\alpha}$ used at the previous time-step. Second order of convergence is a consequence of Theorem 1 below.

4 HOLONOMIC CONSTRAINTS $g(t,y)=0$

We extend here the generalized-α method to systems in mechanics having holonomic constraints $g(t,y)=0$. More precisely we consider

$$M(t,y)y'' = f(t,y,y',\lambda), \quad 0 = g(t,y),$$

where we usually have $f(t,y,y',\lambda) = f_0(t,y,y') - g_y^T(t,y)\lambda$. The term $-g_y^T(t,y)\lambda$ represents reaction forces due to the holonomic constraints $g(t,y)=0$. The algebraic variables λ are associated with the holonomic constraints. Differentiating $0=g(t,y)$ once with respect to t we obtain

$$0 = (g(t,y))' = g_t(t,y) + g_y(t,y)y'.$$

Thus we consider systems of index 2 overdetermined differential-algebraic equations (ODAEs) of the form

$$y' = z, \quad z' = a, \quad 0 = M(t,y)a - f(t,y,z,\lambda), \quad 0 = g(t,y), \quad 0 = g_t(t,y) + g_y(t,y)z,$$

and we assume the matrix

$$\begin{bmatrix} M(t,y) & -f_\lambda(t,y,z,\lambda) \\ g_y(t,y) & O \end{bmatrix} \text{ is invertible .}$$

For $f(t,y,z,\lambda) = f_0(t,y,z) - g_y^T(t,y)\lambda$, this matrix becomes

$$\begin{bmatrix} M(t,y) & g_y^T(t,y) \\ g_y(t,y) & O \end{bmatrix}$$

and is symmetric when $M(t,y)$ is symmetric. At t_0 we consider consistent initial conditions $(y_0, z_0, a_0, \lambda_0)$, i.e.,

$$\begin{aligned} 0 &= M(t_0,y_0)a_0 - f(t_0,y_0,z_0,\lambda_0), \\ 0 &= g(t_0,y_0), \\ 0 &= g_t(t_0,y_0) + g_y(t_0,y_0)z_0, \\ 0 &= g_{tt}(t_0,y_0) + 2g_{ty}(t_0,y_0)z_0 + g_{yy}(t_0,y_0)(z_0,z_0) + g_y(t_0,y_0)a_0. \end{aligned}$$

Several extensions of the HHT-α method have been proposed. Cardona and Géradin in Ref. [1] analyze a direct extension of the HHT-α method to linear DAEs where it was shown that a direct application of the HHT-α method is inconsistent and suffers from instabilities. Yen, Petzold, and Raha in Ref. [8] propose a first order extension of the HHT-α method based on projecting the solution of the underlying ODEs onto the constraints (including the index 1 acceleration level constraints) after each step. More recently, second order extensions of the HHT-α method and generalized-α method have been proposed independently by Jay and Negrut in Ref. [5] and by Lunk and Simeon in Ref. [7] based on the additivity of $f(t,y,z,\lambda) = f_0(t,y,z) + f_1(t,y,\lambda)$. Here, we propose a different and more natural extension of the generalized-α method which does not use this additive structure

$$\begin{aligned}
y_1 &= y_0 + hz_0 + \frac{h^2}{2}\left((1-2\beta)a_\alpha + 2\beta\widetilde{a}_{1+\alpha}\right), \\
z_1 &= z_0 + h\left((1-\gamma)a_\alpha + \gamma a_{1+\alpha}\right), \\
(1-\alpha_m)M_{1+\alpha}\widetilde{a}_{1+\alpha} + \alpha_m M_\alpha a_\alpha &= (1-\alpha_f)f(t_1,y_1,z_1,\widetilde{\lambda}_1) + \alpha_f f(t_0,y_0,z_0,\lambda_0), \quad (11) \\
(1-\alpha_m)M_{1+\alpha}a_{1+\alpha} + \alpha_m M_\alpha a_\alpha &= (1-\alpha_f)f(t_1,y_1,z_1,\lambda_1) + \alpha_f f(t_0,y_0,z_0,\lambda_0), \\
0 &= g(t_1,y_1), \\
0 &= g_t(t_1,y_1) + g_y(t_1,y_1)z_1.
\end{aligned}$$

For $f(t,y,z,\lambda) = f_0(t,y,z) - g_y^T(t,y)\lambda$ we can replace for example Eq. (11) by

$$(1-\alpha_m)M_{1+\alpha}(a_{1+\alpha} - \widetilde{a}_{1+\alpha}) = (1-\alpha_f)g_y^T(t_1,y_1)(\widetilde{\lambda}_1 - \lambda_1).$$

Second order of convergence is a consequence of Theorem 1 below.

5 NONHOLONOMIC CONSTRAINTS $k(t,y,y') = 0$

We extend here the generalized-α method to systems in mechanics having nonholonomic constraints $k(t,y,y') = 0$. More precisely we consider

$$M(t,y)y'' = f(t,y,y',\psi), \quad 0 = k(t,y,y')$$

where we usually have $f(t,y,y',\psi) = f_0(t,y,y') - k_{y'}^T(t,y,y')\psi$. The term $-k_{y'}^T(t,y,y')\psi$ represents reaction forces due to the nonholonomic constraints $k(t,y,y') = 0$. The algebraic variables ψ are associated respectively with the nonholonomic constraints. Hence, we consider systems of index 2 differential-algebraic equations (DAEs) of the form

$$y' = z, \quad z' = a, \quad 0 = M(t,y)a - f(t,y,z,\psi), \quad 0 = k(t,y,z),$$

and we assume the matrix

$$\begin{bmatrix} M(t,y) & -f_\psi(t,y,z,\psi) \\ k_z(t,y,z) & O \end{bmatrix} \text{ is invertible .}$$

For $f(t,y,z,\psi) = f_0(t,y,z) - k_z^T(t,y,z)\psi$, this matrix becomes

$$\begin{bmatrix} M(t,y) & k_z^T(t,y,z) \\ k_z(t,y,z) & O \end{bmatrix}$$

and is symmetric when $M(t,y)$ is symmetric. At t_0 we consider consistent initial conditions (y_0, z_0, a_0, ψ_0), i.e.,

$$\begin{aligned}
0 &= M(t_0,y_0)a_0 - f(t_0,y_0,z_0,\psi_0),\\
0 &= k(t_0,y_0,z_0),\\
0 &= k_t(t_0,y_0,z_0) + k_y(t_0,y_0,z_0)z_0 + k_z(t_0,y_0,z_0)a_0.
\end{aligned}$$

We propose the following extension of the generalized-α method:

$$\begin{aligned}
y_1 &= y_0 + hz_0 + \frac{h^2}{2}\left((1-2\beta)a_\alpha + 2\beta a_{1+\alpha}\right),\\
z_1 &= z_0 + h\left((1-\gamma)a_\alpha + \gamma a_{1+\alpha}\right),\\
(1-\alpha_m)M_{1+\alpha}a_{1+\alpha} + \alpha_m M_\alpha a_\alpha &= (1-\alpha_f)f(t_1,y_1,z_1,\psi_1) + \alpha_f f(t_0,y_0,z_0,\psi_0),\\
0 &= k(t_1,y_1,z_1).
\end{aligned}$$

Second order of convergence is a consequence of Theorem 1 below.

6 GENERAL EXTENSION AND CONVERGENCE

We extend the generalized-α method to systems in mechanics having a nonconstant mass matrix $M(t,y)$, holonomic constraints $g(t,y) = 0$, and nonholonomic constraints $k(t,y,y') = 0$. The algebraic variables λ are associated with the holonomic constraints $g(t,y) = 0$ and $g_t(t,y) + g_y(t,y)y' = 0$ which result from differentiating $g(t,y) = 0$ with respect to t. The algebraic variables ψ are associated with the nonholonomic constraints $k(t,y,y') = 0$. Thus we consider systems of index 2 overdetermined differential-algebraic equations (ODAEs) of the form

$$\begin{aligned}
y' &= z,\\
M(t,y)z' &= f(t,y,z,\lambda,\psi),\\
0 &= g(t,y),\\
0 &= g_t(t,y) + g_y(t,y)z,\\
0 &= k(t,y,z),
\end{aligned} \tag{12}$$

and we assume the matrix

$$\begin{bmatrix} M(t,y) & -f_\lambda(t,y,z,\lambda,\psi) & -f_\psi(t,y,z,\lambda,\psi) \\ g_y(t,y) & O & O \\ k_z(t,y,z) & O & O \end{bmatrix} \text{ is invertible .} \tag{13}$$

For $f(t,y,z,\lambda,\psi) = f_0(t,y,z) - g_y^T(t,y)\lambda - k_z^T(t,y,z)\psi$, this matrix becomes

$$\begin{bmatrix} M(t,y) & g_y^T(t,y) & k_z^T(t,y,z) \\ g_y(t,y) & O & O \\ k_z(t,y,z) & O & O \end{bmatrix}$$

and is symmetric when $M(t,y)$ is symmetric. At t_0 we consider consistent initial conditions $(y_0, z_0, a_0, \lambda_0, \psi_0)$, i.e.,

$$0 = M(t_0,y_0)a_0 - f(t_0,y_0,z_0,\lambda_0,\psi_0),$$

$$
\begin{aligned}
0 &= g(t_0, y_0), \\
0 &= g_t(t_0, y_0) + g_y(t_0, y_0)z_0, \quad 0 = k(t_0, y_0, z_0), \\
0 &= g_{tt}(t_0, y_0) + 2g_{ty}(t_0, y_0)z_0 + g_{yy}(t_0, y_0)(z_0, z_0) + g_y(t_0, y_0)a_0, \\
0 &= k_t(t_0, y_0, z_0) + k_y(t_0, y_0, z_0)z_0 + k_z(t_0, y_0, z_0)a_0.
\end{aligned}
$$

Here, we propose an extension of the generalized-α method which does not use any additive structure of $f(t, y, z, \lambda, \psi)$. We call it the *generalized-α-SOI2 method* (SOI2 stands for Stabilized Overdetermined Index 2). One step $(t_0, y_0, z_0, a_\alpha, \lambda_0, \psi_0) \mapsto (t_1, y_1, z_1, a_{1+\alpha}, \lambda_1, \psi_1)$ with stepsize h of the generalized-α-SOI2 method for Eq. (12) can be expressed as follows

$$
\begin{aligned}
y_1 &= y_0 + hz_0 + \frac{h^2}{2}\left((1-2\beta)a_\alpha + 2\beta\widetilde{a}_{1+\alpha}\right), \\
\widetilde{z}_1 &= z_0 + h\left((1-\gamma)a_\alpha + \gamma\widetilde{a}_{1+\alpha}\right), \\
z_1 &= z_0 + h\left((1-\gamma)a_\alpha + \gamma a_{1+\alpha}\right), \\
(1-\alpha_m)M_{1+\alpha}\widetilde{a}_{1+\alpha} + \alpha_m M_\alpha a_\alpha &= (1-\alpha_f)f(t_1, y_1, z_1, \widetilde{\lambda}_1, \widetilde{\psi}_1) + \alpha_f f(t_0, y_0, z_0, \lambda_0, \psi_0), \\
(1-\alpha_m)M_{1+\alpha}a_{1+\alpha} + \alpha_m M_\alpha a_\alpha &= (1-\alpha_f)f(t_1, y_1, z_1, \lambda_1, \psi_1) + \alpha_f f(t_0, y_0, z_0, \lambda_0, \psi_0), \\
0 &= g(t_1, y_1), \\
0 &= g_t(t_1, y_1) + g_y(t_1, y_1)z_1, \\
0 &= k(t_1, y_1, \widetilde{z}_1), \\
0 &= k(t_1, y_1, z_1),
\end{aligned}
\tag{14}
$$

where $M_{1+\alpha} := M(t_{1+\alpha}, y_0 + h(1+\alpha)z_0)$ and $M_\alpha := M_{(1+\alpha)-1}$ or $M(t_\alpha, y_0 + h\alpha z_0)$ The auxiliary variables $\widetilde{z}_1, \widetilde{a}_{1+\alpha}, \widetilde{\lambda}_1, \widetilde{\psi}_1$ are just local to the current step, they are not propagated. For $f(t, y, z, \lambda, \psi) = f_0(t, y, z) - g_y^T(t, y)\lambda - k_z^T(t, y, z)\psi$ we can replace for example

$$
(1-\alpha_m)M_{1+\alpha}\widetilde{a}_{1+\alpha} + \alpha_m M_\alpha a_\alpha = (1-\alpha_f)f(t_1, y_1, z_1, \widetilde{\lambda}_1, \widetilde{\psi}_1) + \alpha_f f(t_0, y_0, z_0, \lambda_0, \psi_0)
$$

by

$$
(1-\alpha_m)M_{1+\alpha}(a_{1+\alpha} - \widetilde{a}_{1+\alpha}) = (1-\alpha_f)g_y^T(t_1, y_1)(\widetilde{\lambda}_1 - \lambda_1) + (1-\alpha_f)k_z^T(t_1, y_1, z_1)(\widetilde{\psi}_1 - \psi_1).
$$

From Ref. [6] we have the following convergence result:

Theorem 1. *Consider the overdetermined system of DAEs given by Eq. (12) and assumption Eq. (13) with consistent initial conditions $(y_0, z_0, a_0, \lambda_0, \psi_0)$ at t_0 and exact solution $(y(t), z(t), a(t), \lambda(t), \psi(t))$. Suppose that $a_\alpha - a(t_0 + \alpha h) = O(h)$, for example $a_\alpha := a_0$. Then the generalized-α-SOI2 numerical approximation $(y_n, z_n, a_{n+\alpha}, \lambda_n, \psi_n)$ (see Eq. (14)) satisfies for $0 \le h \le h_{\max}$ and $t_n - t_0 = nh \le Const$*

$$
\begin{aligned}
&y_n - y(t_n) = O(h^2), \quad z_n - z(t_n) = O(h^2), \quad a_{n+\alpha} - a(t_n + \alpha h) = O(h^2 + r^n h), \\
&\lambda_n - \lambda(t_n) = O(h^2 + r^n h), \quad \psi_n - \psi(t_n) = O(h^2 + r^n h)
\end{aligned}
$$

where $r := |\alpha_m/(1-\alpha_m)|$. Moreover, if $\alpha_m = 0$ or $a_\alpha - a(t_0 + \alpha h) = O(h^2)$ then we have

$$
a_{n+\alpha} - a(t_n + \alpha h) = O(h^2), \quad \lambda_n - \lambda(t_n) = O(h^2), \quad \psi_n - \psi(t_n) = O(h^2).
$$

7 VARIABLE STEPSIZES h_n

When applying the generalized-α method with variable stepsizes, the values $a_{n+\alpha}$ and $M_{n+\alpha}a_{n+\alpha}$ must be adjusted before each new step in order to preserve the second order of the method. Consider a previous step starting at t_{n-1} with stepsize h_{n-1} and a new step starting at $t_n = t_{n-1} + h_{n-1}$ with stepsize h_n. The value $a_{n-1+\alpha}$ used in the previous step is an approximation of $a(t)$ at $t_{n-1} + \alpha h_{n-1}$ i.e., $a_{n-1+\alpha} \approx a(t_{n-1} + \alpha h_{n-1})$. The value $a_{n+\alpha}$ obtained in the previous step is an approximation of $a(t)$ at $t_{n-1} + (1+\alpha)h_{n-1} = t_n + \alpha h_{n-1}$ i.e., $a_{n+\alpha} \approx a(t_n + \alpha h_{n-1})$. For the current timestep starting at t_n with stepsize h_n we need the value $a_{n+\alpha}$ to be an approximation of $a(t)$ at $t_n + \alpha h_n$, i.e., $a_{n+\alpha} \approx a(t_n + \alpha h_n)$. By linearly interpolating $a_{n-1+\alpha}$ at $t_{n-1} + \alpha h_{n-1}$ and $a_{n+\alpha}$ at $t_n + \alpha h_{n-1}$ and by extrapolating at $t_n + \alpha h_n$, $a_{n+\alpha}$ can be replaced by

$$a_{n+\alpha} \longleftarrow a_{n+\alpha} + \alpha \left(\frac{h_n}{h_{n-1}} - 1 \right) (a_{n+\alpha} - a_{n-1+\alpha}). \tag{15}$$

A similar formula for $M_{n+\alpha}a_{n+\alpha}$ should also be used. We can replace $M_{n+\alpha}a_{n+\alpha}$ by

$$M_{n+\alpha}a_{n+\alpha} \longleftarrow M_{n+\alpha}a_{n+\alpha} + \alpha \left(\frac{h_n}{h_{n-1}} - 1 \right) (M_{n+\alpha}a_{n+\alpha} - M_{n-1+\alpha}a_{n-1+\alpha}). \tag{16}$$

These formulas have several advantages:

- they are simple to implement;
- their computational cost is almost negligible;
- they are valid for both ODEs and DAEs;
- they preserve second order of convergence.

These modifications are not necessary to preserve the second order of convergence for the y and z variables. However, since the cost of these modifications is negligible and they also allow second order of convergence for the a, λ, and ψ variables, these modifications are recommended.

8 NUMERICAL EXAMPLE

To illustrate Theorem 1 numerically we consider the following mathematical test problem

$$\begin{aligned}
&y_1' = z_1, \qquad y_2' = z_2, \\
&\begin{bmatrix} y_1 & y_2 - e^{-2t} \\ \sin(y_1 - e^t) & y_1 y_2 \end{bmatrix} \begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} e^t(y_1 z_2 + 2y_2 z_1) + e^{2t} y_1 \lambda_1 - y_1 z_2 \psi_1 - 2 \\ e^{-t}(y_2 z_2/2 - 2y_1 z_1 y_2 z_2 + y_2 \lambda_1^2) - y_1 y_2 z_1 \psi_1^3 + e^{3t} \end{bmatrix}, \\
&0 = g(t,y) = y_1^2 y_2 - 1, \\
&0 = g_t(t,y) + g_y(t,y)z = 2y_1 y_2 z_1 + y_1^2 z_2, \\
&0 = k(t,y,z) = y_1 z_1 z_2 + 2.
\end{aligned}$$

We have applied the generalized-α-SOI2 method (see Eq. (14)) with damping parameter $\rho_\infty = 0.2$ and variable stepsizes alternating between $h/3$ and $2h/3$ for various values of h. Using the modification of $a_{n+\alpha}$ of Eq. (15) and $M_{n+\alpha}a_{n+\alpha}$ of Eq. (16) we observe global convergence of order 2 at $t_n = 1$ in Fig. 1. Without these modifications in Fig. 2 we observe a reduction of the order of convergence to 1 for the variables a, λ, ψ.

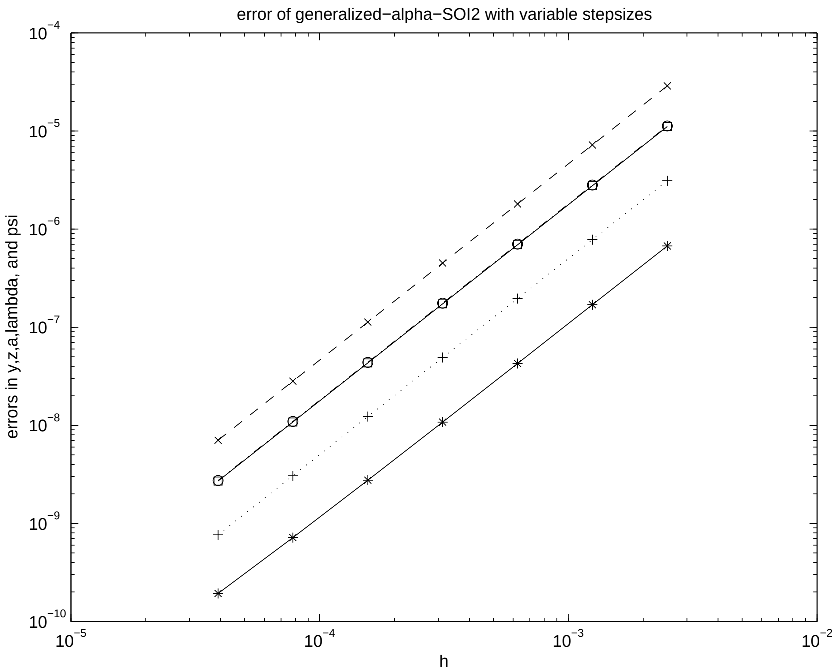

Figure 1: Global errors $\|y_n - y(t_n)\|_2$ ($\square$), $\|z_n - z(t_n)\|_2$ ($\circ$), $\|a_{n+\alpha} - a(t_n + \alpha h)\|_2$ ($\times$), $\|\lambda_n - \lambda(t_n)\|_2$ ($+$), $\|\psi_n - \psi(t_n)\|_2$ ($*$) of the generalized-α-SOI2 method ($\rho_\infty = 0.2$) at $t_n = 1$ for a test problem with variable stepsizes alternating between $h/3$ and $2h/3$ with modification of $a_{n+\alpha}$ of Eq. (15) and $M_{n+\alpha}a_{n+\alpha}$ of Eq. (16).

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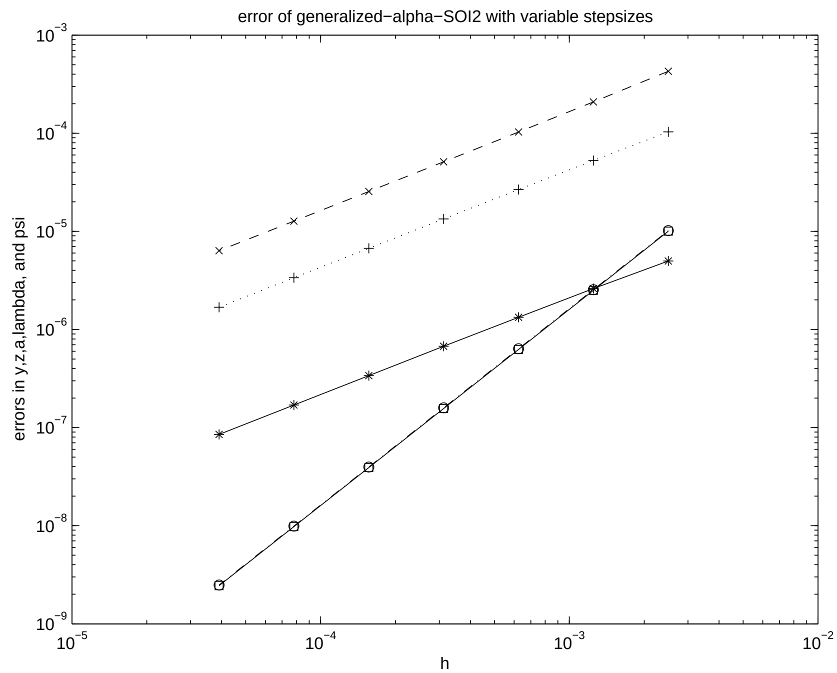

Figure 2: Global errors $\|y_n - y(t_n)\|_2$ ($\Box$), $\|z_n - z(t_n)\|_2$ ($\circ$), $\|a_{n+\alpha} - a(t_n + \alpha h)\|_2$ ($\times$), $\|\lambda_n - \lambda(t_n)\|_2$ ($+$), $\|\psi_n - \psi(t_n)\|_2$ ($*$) of the generalized-α-SOI2 method ($\rho_\infty = 0.2$) at $t_n = 1$ for a test problem with variable stepsizes alternating between $h/3$ and $2h/3$ without modification of $a_{n+\alpha}$ of Eq. (15) and $M_{n+\alpha}a_{n+\alpha}$ of Eq. (16).