Resonance sums for Rankin–Selberg products of \( SL_m(\mathbb{Z}) \) Maass cusp forms

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Let \( f \) and \( g \) be Maass cusp forms for \( SL_m(\mathbb{Z}) \) and \( SL_{m'}(\mathbb{Z}) \), respectively, with \( 2 \leq m \leq m' \). Let \( \lambda_{f \times g}(n) \) be the normalized coefficients of \( L(s, f \times g) \), the Rankin–Selberg L-function for \( f \) and \( g \). In this paper the asymptotics of a Voronoi-type summation formula for \( \lambda_{f \times g}(n) \) are derived. As an application estimates are obtained for the smoothly weighted average of \( \lambda_{f \times g}(n) \) against \( e(\alpha n^\beta) \). When \( \beta = \frac{1}{mm'} \) and \( \alpha \) is close or equal to \( \pm mm'q \frac{1}{mm'} \) for a positive integer \( q \), the average has a main term of size \( |\lambda_{f \times g}(q)|X \frac{1}{mm'} + \frac{1}{2} \). Otherwise, when \( 0 < \beta < \frac{1}{mm'} \), it is shown that this average decays rapidly. This phenomenon is due to the oscillatory nature of the coefficients \( \lambda_{f \times g}(n) \).

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1. Introduction

Consider the sum

\[
\sum_{n=1}^{\infty} \lambda_f(n) \phi \left( \frac{n}{X} \right) e(\alpha n^\beta)
\]

(1)
where $\alpha, \beta, X \in \mathbb{R}$ are parameters to be specified later, $\phi(x)$ is some compactly supported function, the $\lambda_f(n)$ are coefficients attached to an object of interest, and $e(x) = e^{2\pi i x}$ as usual. The goal is to examine the oscillatory behavior of the coefficients $\lambda_f(n)$. One way this is accomplished is through the study of weighted sums of these coefficients against various exponential functions whose oscillatory behavior is well known. The constructive and destructive interferences of these oscillations give rise to resonance and decay, respectively. Consequently, we refer to the summation in equation (1) as a resonance sum.

We now present a brief summary on what is currently known regarding resonance sums for convenience. For additional results and applications we refer the reader to [20, 26,25,27,28].

- When $f$ is an automorphic representation of $GL_m(A)\mathbb{Q}$ and $\lambda_f(n)$ are the coefficients of the associated $L$-function, Booker [2] showed that (1) has rapid decay for $\alpha = 0$.
- When $f$ is a $SL_2(\mathbb{Z})$ cusp form and $\lambda_f(n)$ are the coefficients of the associated $L$-function (or, equivalently, the Fourier coefficients), Iwaniec et al. [10] and then Ren and Ye [19] gave asymptotics and upper bounds of (1) for $\alpha \neq 0$.
- When $f$ is a $SL_3(\mathbb{Z})$ Maass (cusp) form and $\lambda_f(n)$ are the coefficients of the associated $L$-function (or, equivalently, the Fourier coefficients), Ernvall-Hytönen [3] and later Ren and Ye [22,23] gave asymptotics and upper bounds of (1) for $\beta = 1$ and $\alpha, \beta \geq 0$, respectively.
- When $f$ is a $SL_m(\mathbb{Z})$ Maass cusp form and $\lambda_f(n)$ are the coefficients of the associated $L$-function (or, equivalently, the Fourier–Whittaker coefficients), Ren and Ye [21] gave asymptotics and upper bounds of (1) for $\alpha, \beta \geq 0$. Concurrently and independently, Ernvall-Hytönen et al. [4] gave asymptotics and $\Omega$-results of (1) for $\beta = 1$.
- When $f$ is an element of degree $d$ of the extended Selberg class of functions and $\lambda_f(n)$ are the coefficients of the associated $L$-function, Kaczorowski and Perelli [12] gave asymptotics of (1) for $\alpha > 0$ and $\beta = \frac{1}{d}$ under various hypotheses.

In this paper we continue these investigations and describe the asymptotics in the case when $\lambda_{f \times g}(n)$ are the coefficients of the Rankin–Selberg $L$-function attached to $f \times g$ where $f$ and $g$ are Maass cusp forms of $SL_m(\mathbb{Z})$ and $SL_{m'}(\mathbb{Z})$, respectively, with $2 \leq m \leq m'$ and $\alpha, \beta \geq 0$. First, we derive the following summation formula.

**Theorem 1.** Let $f$ and $g$ be Maass cusp forms for $SL_m(\mathbb{Z})$ and $SL_{m'}(\mathbb{Z})$, respectively, with $2 \leq m \leq m'$. If $m = m'$ then suppose that $f$ and $g$ are not twist equivalent. Let $\psi(x) \in C^\infty(0, \infty)$ with compact support, then

$$
\sum_{n=1}^{\infty} \lambda_{f \times g}(n)\psi(n) = 2\pi \int_{0}^{\infty} \psi(x)G_{f \times g}(\frac{\pi x^2}{4})dx
$$

(2)
where \( \ell = mm' \) and

\[
G_{\ell \times g}(x) = G_{0,2\ell}^{\ell,0} \left( \begin{array}{c}
-\frac{\mu_f(j) + \mu_g(k)}{2}, & 1 + \frac{\mu_f(j) + \mu_g(k)}{2}
\end{array} \bigg| x \right).
\]

Here \( G_{0,2\ell}^{\ell,0} \) is the Meijer G-function, \( \mu \) are the parameters at \( \infty \), and it is understood that the indices \( j \) and \( k \) take all possible values.

The particular choice of \( \psi(x) = e(\alpha x^\beta) \phi \left( \frac{x}{X} \right) \) in equation (2) gives an exact formula for the resonance sum under consideration. With this choice of \( \psi \) we refer to equation (2) as a Voronoi-type summation formula because it is similar to the full Voronoi summation formulas of Goldfeld and Li [6,7] and of Miller and Schmid [17,18]. In fact, Theorem 1 and Theorem 5.1 of [6] are similar and can be derived from one another (see Remark 3 closing §3.1). This Voronoi-type summation formula, together with the known asymptotics of the Meijer-G function, yields the following theorem.

**Theorem 2.** Let \( f \) and \( g \) be Maass cusp forms for \( SL_m(\mathbb{Z}) \) and \( SL_{m'}(\mathbb{Z}) \), respectively, with \( 2 \leq m \leq m' \). If \( m = m' \) then suppose that \( f \) and \( \tilde{g} \) are not twist equivalent. Let \( \phi(x) \in C^\infty(0, \infty) \) with compact support in \([1, 2]\) and \( \phi^{(j)}(x) \ll 1 \) for \( j \geq 1 \). Moreover, let \( mm' = \ell \), \( X > 1 \), and \( \alpha, \beta \geq 0 \).

(i) If \( 2 \max \{1, 2^{\beta - \frac{1}{2}}\}(\alpha \beta)^{\ell} \leq X^{1 - \beta \ell} \), then

\[
\sum_{n=1}^{\infty} \lambda_{f \times g}(n) \phi \left( \frac{n}{X} \right) e(\pm \alpha n^\beta) \ll_{\ell, \beta, M} X^{-M}
\]

holds for any \( M > 0 \).

(ii) If \( 2 \max \{1, 2^{\beta - \frac{1}{2}}\}(\alpha \beta)^{\ell} > X^{1 - \beta \ell} \), then

\[
\sum_{n=1}^{\infty} \lambda_{f \times g}(n) \phi \left( \frac{n}{X} \right) e(\pm \alpha n^\beta) \ll_{\ell, \beta} (\alpha X^\beta)^{\frac{\ell}{2}}
\]

holds for \( \beta \neq \frac{1}{\ell} \), and

\[
\sum_{n=1}^{\infty} \lambda_{f \times g}(n) \phi \left( \frac{n}{X} \right) e(\pm \alpha n^\beta) \ll_{\ell, \beta} (\alpha X^\beta)^{\frac{1+\ell}{2}}
\]

holds for \( \beta = \frac{1}{\ell} \).

(iii) If \( X > \alpha^{\frac{\ell-1}{1-\ell \varepsilon}} \) with \( 0 < \varepsilon < \frac{1}{\ell} \), then

\[
\sum_{n=1}^{\infty} \lambda_{f \times g}(n) \phi \left( \frac{n}{X} \right) e(\pm \alpha n^\beta) = \frac{\lambda_{f \times g}(n_\alpha)}{n_\alpha} \sum_{k=0}^{r} c_{k, \ell}(n_\alpha; -)(n_\alpha X)\frac{\beta - \frac{1}{2} - \frac{\ell}{2}}{2}
\]

\[
+ O_{\ell, \varepsilon} \left( X^{\frac{\beta - \frac{1}{2} - \frac{\ell}{2}}{2}} \right)
\]
for any $r > \frac{\ell - 1}{2}$. Here $n_{\alpha}$ is the unique positive integer satisfying $\left(\frac{\alpha}{\ell}\right) \ell - n_{\alpha} \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

\[ I_k(n_{\alpha}; -) = \int_0^\infty \phi(t^\ell) e\left(\left(\alpha - n_{\alpha} \frac{i}{\sqrt{\ell}}\right) X^{\frac{1}{2}} t\right) t^{\frac{s}{2} - \frac{r}{2} - k} dt, \]

and $c_{k,\ell}$ are constants depending on $f$, $g$, $k$, and $\ell$.

(iv) In particular, if $q$ is a positive integer and $0 < \varepsilon < \frac{1}{\ell}$, then for $X > (\ell^\ell q)^{\frac{\ell - 1}{2\ell}}$ we have

\[ \sum_{n=1}^\infty \lambda_{f \times \tilde{g}}(n) \phi\left(\frac{n}{X}\right) e(\pm \ell(qn)^{\frac{1}{2}}) = \frac{\lambda_{f \times \tilde{g}}(q)}{q} \sum_{k=0}^r c_{k,\ell} I_k(q; -)(qX)^{\frac{1}{2\ell} - \frac{1}{2} - \frac{k}{2}} + O_{\ell, r, \varepsilon} \left(X^{\frac{1}{2\ell} - \frac{1}{2} - \frac{r + 1}{2}}\right) \]

for any $r > \frac{\ell - 1}{2}$ where

\[ I_k(q; -) = \frac{1}{\ell} \int_0^\infty \phi(x) x^{\frac{1}{2\ell} - \frac{r}{2} - \frac{k}{2}} dx \]

and $c_{k,\ell}$ are constants depending on $f$, $g$, $k$, and $\ell$.

All proofs will be given in Section 3 following some preliminaries on Maass forms and the Meijer $G$-function in Section 2. We conclude this section with a few remarks about these results.

**Remark 1.** The condition that $f$ and $\tilde{g}$ not be twist equivalent may be dropped in both Theorems 1 and 2 with appropriate modifications. Indeed, the only complication of this case is the possible existence of poles in the completed $L$-function which will lead to a residual term, say $R$, in equation (2). It can be shown that $R \ll \ell, f, g \int_0^\infty e(\alpha x^\beta) \phi\left(\frac{x}{X}\right) dx$ which can then be made negligible via repeated integration by parts.

**Remark 2.** Analogues of Theorems 1 and 2 hold for individual Maass cusp forms with appropriate modifications. Consequently, the $L$-function coefficients (and therefore the Fourier–Whittaker coefficients) of the Rankin–Selberg product between a $SL_m(\mathbb{Z})$ form and a $SL_{m'}(\mathbb{Z})$ form exhibit the same behavior as those of a $SL_{mm'}(\mathbb{Z})$ form. This provides indirect evidence for functoriality in that Theorem 2 is precisely what one would expect given the resonance of Maass cusp forms.
2. Preliminaries

2.1. Maass forms

Following Chapters 5 and 12 of Goldfeld [5], let \( f \) and \( g \) be Maass cusp forms for \( SL_m(\mathbb{Z}) \) and \( SL_m'(\mathbb{Z}) \), respectively. Throughout Sections 2 and 3 we assume \( 2 \leq m \leq m' \) where \( f \) and \( \tilde{g} \) are not twist equivalent if \( m = m' \). The associated \( L \)-functions are

\[
L(s, f) := \prod_p \prod_{j=1}^{m}(1 - \alpha_{p,j}p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}
\]

and

\[
L(s, g) := \prod_p \prod_{k=1}^{m'}(1 - \beta_{p,k}p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^s},
\]

where \( \lambda_f(n) = A_f(n,1,\ldots, 1) \) and \( \lambda_g(n) = A_g(n,1,\ldots, 1) \in \mathbb{C} \) are the normalized Fourier–Whittaker coefficients, and \( \alpha_{p,j}, \beta_{p,k} \in \mathbb{C} \) are the Satake parameters at \( p \). The completed \( L \)-functions are given by

\[
\Lambda(s, f) := \pi^{-\frac{m}{2}} \prod_{j=1}^{m} \Gamma \left( \frac{s - \mu_f(j)}{2} \right) L(s, f) \quad \text{and}
\]

\[
\Lambda(s, g) := \pi^{-\frac{m'}{2}} \prod_{k=1}^{m'} \Gamma \left( \frac{s - \mu_g(k)}{2} \right) L(s, g),
\]

where \( \mu_f(j) \) and \( \mu_g(k) \) are the parameters at \( \infty \). The completed \( L \)-functions are entire and satisfy the functional equations \( \Lambda(s, f) = \Lambda(1-s, \tilde{f}) \) and \( \Lambda(s, g) = \Lambda(1-s, \tilde{g}) \), where \( \tilde{f} \) and \( \tilde{g} \) are the dual Maass forms.

Given the Maass forms \( f \) and \( g \) we can form the Rankin–Selberg product \( f \times g \). The Rankin–Selberg product also has an associated \( L \)-function which is given by

\[
L(s, f \times \tilde{g}) := \prod_p \prod_{j=1}^{m} \prod_{k=1}^{m'} (1 - \alpha_{p,j}\beta_{p,k}p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_{f \times \tilde{g}}(n)}{n^s}.
\]

Initially the \( L \)-function converges absolutely for \( \Re(s) \gg 1 \), but ultimately [11] we get absolute convergence for \( \Re(s) > 1 \). The completed \( L \)-function

\[
\Lambda(s, f \times \tilde{g}) := \pi^{-\frac{mm'}{2}} \prod_{j=1}^{m} \prod_{k=1}^{m'} \Gamma \left( \frac{s - \mu_f(j) - \mu_g(k)}{2} \right) L(s, f \times \tilde{g})
\]

extends to an entire function via the functional equation \( \Lambda(s, f \times \tilde{g}) = \Lambda(1-s, \tilde{f} \times g) \). Note that from the functional equation we have
\[ L(s, f \times \tilde{g}) = \pi^{m'm'} \prod_{j=1}^{m} \prod_{k=1}^{m'} \frac{\Gamma \left( \frac{1-s-\mu(j)-\mu(k)}{2} \right)}{\Gamma \left( \frac{s-\mu(j)-\mu(k)}{2} \right)} L(1-s, \tilde{f} \times g). \] (3)

2.2. Meijer G-function

The main reference for this subsection is Chapter 5 of Luke [15,16]. The Meijer G-function is a generalized function which encompasses many special functions. It was originally defined through a series representation, but it is now more commonly given by the following inverse Mellin transform.

**Definition 1.** The Meijer G-function is defined as

\[ G^{m,n}_{p,q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| z \right) = \frac{1}{2\pi i} \int_L \prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1-a_j + s) ds \]

where \( 0 \leq m \leq q \) and \( 0 \leq n \leq p \) are integers, the complex parameters \( a_k \) and \( b_j \) are such that no pole of \( \Gamma(b_j - s) \), \( j = 1, \ldots, m \), coincides with any pole of \( \Gamma(1-a_k + s) \), \( k = 1, \ldots, n \), and \( z \neq 0 \). The path of integration may be chosen as follows:

(i) \( L \) goes from \( \sigma - i\infty \) to \( \sigma + i\infty \) so that all the poles of \( \Gamma(b_j - s) \), \( j = 1, \ldots, m \), lie to the right of the path, and all the poles of \( \Gamma(1-a_k + s) \), \( k = 1, \ldots, n \), lie to the left of the path. For the integral to converge we need \( \delta = m + n - \frac{1}{2}(p + q) > 0 \), \( \left| \text{arg}(z) \right| < \delta \pi \). If \( \left| \text{arg}(z) \right| = \delta \pi \), \( \delta \geq 0 \), the integral converges absolutely when \( p = q \) if \( \Re(\nu) < -1 \); and when \( p \neq q \), if with \( s = \sigma + i\tau \), \( \sigma \) is chosen so that for \( \tau \to \pm\infty \),

\[ (q-p)\sigma > \Re(\nu) + 1 - \frac{1}{2}(q-p), \]

\[ \nu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j. \]

(ii) \( L \) is a loop beginning and ending at \( +\infty \) and encircling all poles of \( \Gamma(b_j - s) \), \( j = 1, \ldots, m \), once in the negative direction, but none of the poles of \( \Gamma(1-a_k + s) \), \( k = 1, \ldots, n \). The integral converges if \( q \geq 1 \) and either \( p < q \) or \( p = q \) and \( |z| < 1 \).

(iii) \( L \) is a loop beginning and ending at \( -\infty \) and encircling all poles of \( \Gamma(1-a_k + s) \), \( k = 1, \ldots, n \), once in the positive direction, but none of the poles of \( \Gamma(b_j - s) \), \( j = 1, \ldots, m \). The integral converges if \( p \geq 1 \) and either \( p > q \) or \( p = q \) and \( |z| > 1 \).

When the parameters \( a_k \) and \( b_j \) are clear the Meijer G-function is usually denoted as \( G^{m,n}_{p,q} \left( \begin{array}{c} a_k \\ b_j \end{array} \middle| z \right) \) or \( G^{m,n}_{p,q}(z) \). We note that in our applications we will have \( \text{arg}(z) = 0 \) and consequently will take the first contour described above. The Meijer G-function satisfies
many interesting properties including multiplication formulas and differential equations. In this paper, however, we will only require the following asymptotic expansion.

**Lemma 1.** For integers \( m \geq 1 \) and \( r \geq 0 \) we have

\[
G_{0,2m}^{m,0}(\frac{-b_j}{x}) = A_{2m}^{m,0}H_{0,2m}(xe^{-i\pi m};r) + \bar{A}_{2m}^{m,0}H_{0,2m}(xe^{i\pi m};r) \tag{5}
\]

as \( x \to +\infty \) where

\[
A_{2m}^{m,0} = \left(-\frac{1}{2\pi i}\right)^m \exp\left(-i\pi \sum_{j=m+1}^{2m} b_j\right),
\]

\[
H_{0,2m}(x; r) = \frac{(2\pi)^m}{2\sqrt{\pi m}} \exp(-2mx^{\frac{1}{2m}}) \left(x^\theta \sum_{k=0}^{r} M_k x^{-\frac{k}{2m}} + O\left(x^{\theta - \frac{r+1}{2m}}\right)\right),
\]

\[
\theta = \frac{1}{4m} \left(1 - 2m + 2 \sum_{j=1}^{2m} b_j\right),
\]

and the \( M_k \)'s are constants, independent of \( x \), depending on \( m \) and \( b_j \).

**Proof.** The asymptotic expansions of the general \( G_{p,q}^{m,n}(z) \) are given in [15,16]. For our particular choice of \( m, n, q, \) and \( q \) see equation (4) in Theorem 2 of section 5.9.2, equation (2) of section 5.8.2, and Theorem 5 of section 5.9.1 in [16]. □

3. Proof of results

3.1. Proof of Theorem 1

We proceed as in [6,9,13] to deduce the associated summation formula. Let

\[
\mathcal{M}(A)(s) := \int_0^\infty A(x)x^{s-1}dx \quad \text{and} \quad \mathcal{M}^{-1}(B)(x) := \frac{1}{2\pi i} \int_{(\sigma)} B(s)x^{-s}ds,
\]

denote the Mellin and inverse Mellin transforms, respectively, and let \( \psi(x) \) be a test function satisfying the hypotheses of the theorem. Multiplying both sides of (3) by \( \frac{1}{2\pi i} \mathcal{M}(\psi)(s) \) and integrating along the line \( \Re(s) = 1 + \varepsilon \) for some \( \varepsilon > 0 \) we have

\[
\frac{1}{2\pi i} \int_{(1+\varepsilon)} L(s, f \times \tilde{g})\mathcal{M}(\psi)(s)ds = \frac{1}{2\pi i} \int_{(1+\varepsilon)} \pi^{mm'} s^{-\frac{mm'}{2}} \mathcal{M}(\psi)(s)
\]
\[
\times \prod_{j=1}^{m} \prod_{k=1}^{m'} \frac{\Gamma\left(\frac{1-s-\mu_j(j)-\mu_k(k)}{2}\right)}{\Gamma\left(\frac{s-\mu_j(j)-\mu_k(k)}{2}\right)} L(1-s, \tilde{f} \times g) ds.
\]

(6)

Note that the integration is well defined since the \(L\)-function has at most polynomial growth in vertical strips of \(\Re(s) > 0\) and \(\mathcal{M}(\psi)(s)\) is of rapid decay as \(\Im(s) \to \pm \infty\). We evaluate the left hand side of (6) by interchanging the summation and integration, which is valid by Lemma 2 below, to obtain the sum

\[
\sum_{n=1}^{\infty} \lambda_{f \times \tilde{g}}(n) \psi(n).
\]

To evaluate the right hand side of (6) we shift the contour left to \(\Re(s) = -1\) and make the change of variable \(s \mapsto 1-s\) to get

\[
\frac{1}{2\pi i} \int (2) \pi^{-mm's + \frac{mm'}{2}} \mathcal{M}(\psi)(1-s) \prod_{j=1}^{m} \prod_{k=1}^{m'} \frac{\Gamma\left(\frac{s-\mu_j(j)-\mu_k(k)}{2}\right)}{\Gamma\left(\frac{1-s-\mu_j(j)-\mu_k(k)}{2}\right)} L(s, \tilde{f} \times g) ds.
\]

Observe that the integrand has no poles because it is equal to \(L(1-s, f \times \tilde{g}) \mathcal{M}(\psi)(1-s)\) which is entire when \(f\) and \(\tilde{g}\) are not twist equivalent. Also, the residual horizontal integrals vanish since \(\mathcal{M}(\psi)\) has rapid decay. Putting in the series for \(L(s, \tilde{f} \times g)\) and making the change of variables \(s \mapsto 2s\) we have

\[
2\pi \frac{mm'}{2\pi i} \int (1) \sum_{n=1}^{\infty} \lambda_{f \times \tilde{g}}(n) \left(\pi^{2mm' n^2}\right)^{-s} \mathcal{M}(\psi)(1-2s)
\]

\[
\times \prod_{j=1}^{m} \prod_{k=1}^{m'} \frac{\Gamma\left(s - \frac{\mu_j(j)+\mu_k(k)}{2}\right)}{\Gamma\left(1-s - \frac{1+\mu_j(j)+\mu_k(k)}{2}\right)} ds.
\]

(7)

To interchange the summation and integration we need the following two lemmas.

**Lemma 2.** Suppose \(\{D(n, s)\}\) is a sequence of complex measurable functions defined almost everywhere on \(\Re(s) = \sigma_0\) such that

\[
\sum_{n=1}^{\infty} \int_{(\sigma_0)} |D(n, s)| ds < \infty.
\]

Then we have

\[
\int_{(\sigma_0)} \sum_{n=1}^{\infty} D(n, s) ds = \sum_{n=1}^{\infty} \int_{(\sigma_0)} D(n, s) ds.
\]
Proof. This is Theorem 1.38 in Rudin [24]. □

Lemma 3. Let $D(n,s)$ be given by

$$D(n,s) := \lambda_{\hat{f} \times g}(n) \left( \frac{\pi^{2mm'}}{n^2} \right)^{−s} \mathcal{M}(\psi)(1−2s) \prod_{j=1}^{m} \prod_{k=1}^{m'} \frac{\Gamma \left( s - \frac{\mu_f(j) + \mu_g(k)}{2} \right)}{\Gamma \left( 1 - s - \frac{1+\mu_f(j) + \mu_g(k)}{2} \right)},$$

then it follows that

$$\int \sum_{n=1}^{\infty} D(n,s)ds = \sum_{n=1}^{\infty} \int D(n,s)ds$$

whenever $L(2s, \hat{f} \times g)$ converges absolutely.

Proof. By Lemma 2 it suffices to show that

$$\sum_{n=1}^{\infty} \int |D(n,s)|ds < \infty$$

whenever $L(2s, \hat{f} \times g)$ converges absolutely. For $s = \sigma_0 + it$ we have

$$|D(n,s)| = |\lambda_{\hat{f} \times g}(n)| \left( \frac{\pi^{2mm'}}{n^2} \right)^{−\sigma_0} |\mathcal{M}(\psi)(1−2\sigma_0 − 2it)| \prod_{j=1}^{m} \prod_{k=1}^{m'} \frac{\left| \Gamma \left( \sigma_0 + it - \frac{\mu_f(j) + \mu_g(k)}{2} \right) \right|}{\left| \Gamma \left( 1 - \sigma_0 - it - \frac{1+\mu_f(j) + \mu_g(k)}{2} \right) \right|}.$$

First, we claim that

$$\prod_{j=1}^{m} \prod_{k=1}^{m'} \left| \frac{\Gamma \left( \sigma_0 + it - \frac{\mu_f(j) + \mu_g(k)}{2} \right)}{\Gamma \left( 1 - \sigma_0 - it - \frac{1+\mu_f(j) + \mu_g(k)}{2} \right) \right| = O \left( |t|^{\frac{mm'}{2}(2\sigma_0 - \frac{1}{2})} \right).$$

By Stirling’s Formula (eq. (6.1.45) in [1]) we know

$$\left| \Gamma(\sigma + it) \right| \sim \sqrt{2\pi} e^{-\frac{\pi|t|}{2}} |t|^\sigma \frac{1}{2}$$

as $t \to \pm \infty$. Letting $\mu_f(j) = \rho_f(j) + i\tau_f(j)$, and similarly for $\mu_g(k)$, we have

$$\left| \Gamma \left( \sigma_0 + it - \frac{\mu_f(j) + \mu_g(k)}{2} \right) \right| \sim \sqrt{2\pi} e^{-\frac{\pi|t|}{2}} \left| t + \frac{\tau_f(j) - \tau_g(k)}{2} \right|^\sigma \frac{1}{2}$$

$$\sim \sqrt{2\pi} e^{-\frac{\pi|t|}{2}} \left| t + \frac{\tau_f(j) - \tau_g(k)}{2} \right|^\sigma \frac{1}{2}$$
and similarly
\[
\left| \Gamma \left( \frac{1}{2} - \sigma_0 - it - \frac{\mu_f(j) + \mu_g(k)}{2} \right) \right| \\
\sim \sqrt{2\pi e^{-\frac{(\mu_f(j) - \mu_g(k))^2}{2}}} \left| t + \frac{\mu_f(j) - \mu_g(k)}{2} \right|^{-\sigma_0 - \frac{\mu_f(j) + \mu_g(k)}{2}}
\]
so we have
\[
\frac{\left| \Gamma \left( \sigma_0 + it - \frac{\mu_f(j) + \mu_g(k)}{2} \right) \right|}{\left| \Gamma \left( 1 - \sigma_0 - it - \frac{1 + \mu_f(j) + \mu_g(k)}{2} \right) \right|} \sim \left| t + \frac{\mu_f(j) - \mu_g(k)}{2} \right|^{2\sigma_0 - \frac{1}{2}}.
\]
Consequently
\[
\frac{\left| \Gamma \left( \sigma_0 + it - \frac{\mu_f(j) + \mu_g(k)}{2} \right) \right|}{\left| \Gamma \left( 1 - \sigma_0 - it - \frac{1 + \mu_f(j) + \mu_g(k)}{2} \right) \right|} = O \left( |t|^{2\sigma_0 - \frac{1}{2}} \right)
\]
and the claim follows.

Next we show that
\[
I = \int_{-\infty}^{\infty} |M(\psi)(1 - 2\sigma_0 - 2it)| \prod_{j=1}^{m} \prod_{k=1}^{m'} \left| \frac{\Gamma \left( \sigma_0 + it - \frac{\mu_f(j) + \mu_g(k)}{2} \right)}{\Gamma \left( 1 - \sigma_0 - it - \frac{1 + \mu_f(j) + \mu_g(k)}{2} \right)} \right| dt < \infty.
\]
Let \( B \in (0, \infty) \) and split up the integral into three pieces: \( I_-, I_0, \) and \( I_+ \) over \((-\infty, -B), (-B, B), \) and \((B, \infty), \) respectively. Since \( I_0 < \infty \) it remains to show \( I_+ < \infty. \) Here we choose \( B \) sufficiently large so that the claim above yields
\[
I_+ = \int_{B}^{\infty} \left| M(\psi)(1 - 2\sigma_0 - 2it) \right| \prod_{j=1}^{m} \prod_{k=1}^{m'} \left| \frac{\Gamma \left( \sigma_0 + it - \frac{\mu_f(j) + \mu_g(k)}{2} \right)}{\Gamma \left( 1 - \sigma_0 - it - \frac{1 + \mu_f(j) + \mu_g(k)}{2} \right)} \right| dt
\]
\[
\ll \int_{B}^{\infty} \left| M(\psi)(1 - 2\sigma_0 - 2it) \right| \left| t \right|^{m'm'(2\sigma_0 - \frac{1}{2})} dt < \infty
\]
since \( M(1 - 2s) \) is of rapid decay. A similar argument shows that \( I_- < \infty \) and hence \( I < \infty. \) Thus,
\[
\sum_{n=1}^{\infty} \int_{\sigma_0}^{\infty} \left| D(n, s) \right| ds = I \left| \pi^{2mm'} \right| - \sigma_0 \sum_{n=1}^{\infty} \frac{\left| \lambda_{\bar{f} \times g}(n) \right|}{n^{2\sigma_0}} < \infty
\]
wherever \( L(2s, \bar{f} \times g) \) converges absolutely. \( \square \)
Thus, we may interchange the order of the summation and integration in \( (7) \) wherever \( L(2s, \tilde{f} \times g) \) converges absolutely; i.e. for \( \Re(s) > \frac{1}{2} \). Replacing \( s \mapsto -s \) in \( (7) \) we have

\[
2\pi \frac{m' m}{2} \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{C} \lambda_{\tilde{f} \times g}(n) \left( \pi^{2mm'} n^2 \right)^{s} \mathcal{M}(\psi)(1 + 2s) \rho \int_{0}^{\infty} \psi(x) \left( \pi^{2mm'} x^2 n^2 \right)^{s} \prod_{j=1}^{m} \prod_{k=1}^{m'} \frac{\Gamma(-s - \frac{\mu_{f}(j) + \mu_{g}(k)}{2})}{\Gamma(1 + s - \frac{1 + \mu_{f}(j) + \mu_{g}(k)}{2})} ds \text{d}x.
\]

Note that at this point a suitable change of variables would yield the Voronoi-type summation formula given in Theorem 5.1 of [6] (see Remark 3 below).

The next step is to replace the Mellin transform with its integral and interchange the two integrals. To do so we require the absolute convergence of the resulting integral over \( s \) which will be a Meijer \( G \)-function. This integral will converge absolutely provided we take the contour \( L \) such that all of the poles of \( \Gamma\left(-s - \frac{\mu_{f}(j) + \mu_{g}(k)}{2}\right) \) lie to the right of the contour \( L \), and \( s = \sigma + i\tau \in L \) satisfies \( \sigma > \frac{1}{2\ell} - \frac{1}{4} \) as \( \tau \to \pm \infty \). Observe that the poles of the Gamma function occur in the right half-plane \( \Re(s) \geq -\frac{1}{2} \) by the trivial bound of the parameters at infinity (this can be improved to \( \Re(s) \geq 0 \) if one has the Selberg eigenvalue conjecture). Moreover, this contour deformation is justified because the integrand decays rapidly as \( |t| \to \infty \).

Hence, putting in the integral expression for \( \mathcal{M}(\psi)(1+2s) \), deforming the contour if necessary, and interchanging the integrals we obtain

\[
\int_{0}^{\infty} \psi(x) \left( \pi^{2mm'} x^2 n^2 \right)^{s} \prod_{j=1}^{m} \prod_{k=1}^{m'} \frac{\Gamma(-s - \frac{\mu_{f}(j) + \mu_{g}(k)}{2})}{\Gamma(1 + s - \frac{1 + \mu_{f}(j) + \mu_{g}(k)}{2})} ds \text{d}x.
\]

Rewriting this as a Meijer \( G \)-function and equating the right and left hand sides of equation \( (6) \) we obtain the summation formula

\[
\sum_{n=1}^{\infty} \lambda_{\tilde{f} \times g}(n) \psi(n) = 2\pi \frac{m' m}{2} \sum_{n=1}^{\infty} \lambda_{\tilde{f} \times g}(n) \int_{0}^{\infty} \psi(x) G_{\tilde{f} \times g}(\pi^{2\ell} x^2 n^2) dx
\]

given in equation \( (2) \) where \( \ell = mm' \) and

\[
G_{\tilde{f} \times g}(x) = G_{0,2\ell}^{\ell,0} \left( \begin{array}{c}
\frac{\mu_{f}(j) + \mu_{g}(k)}{2}

\frac{1 + \mu_{f}(j) + \mu_{g}(k)}{2}

\end{array} | x \right).
\]

Here it is understood that the indices \( j \) and \( k \) take all possible values.

**Remark 3.** The explicit relationship between Theorem 1 and Theorem 5.1 in [6] of Goldfeld and Li is as follows. First, note the difference in the definitions of the Rankin–Selberg
Thus, the and valid and restated by shows this of function: $L(s, g \times \tilde{f}) = L_g \times_f (s)$. The change of variables $s \mapsto 1 + 2s$ in equation (5.12) of [6] shows that $\Phi(n)/n$ is equal to $\lambda_{g \times f}(n)$ times the expression in equation (8). Using this equality and setting $n = m_1 m_2^{-1} \ldots m_t$, so that

$$
\lambda_{g \times f}(n) = \sum_{n = m_1 m_2^{-1} \ldots m_t} B_g(m_2, \ldots, m_t) A_f(1, \ldots, 1, m_1, \ldots, m_t),
$$

shows that the summation formula in equation (5.14) of [6] is equivalent to that obtained by concluding the proof at equation (8) above. In particular, Theorems 1 and 2 may be restated explicitly in terms of the Fourier–Whittaker coefficients $A_f$ and $B_g$.

### 3.2. Proof of Theorem 2

We begin by setting

$$
\psi(x) := e(\alpha x^\beta) \phi \left( \frac{x}{X} \right)
$$

and applying Lemma 1 to equation (9) to obtain the asymptotic expansion

$$
G_{\tilde{f} \times g}(x) = A^{\ell,0}_{2\ell} H_{0,2\ell}(xe^{-i\pi \ell}; r) + \tilde{A}^{\ell,0}_{2\ell} H_{0,2\ell}(xe^{i\pi \ell}; r)
$$

valid for any integer $r \geq 0$ and real $x \to +\infty$ where

$$
A^{\ell,0}_{2\ell} = \left( -\frac{1}{2\pi i} \right)^\ell \exp \left( i\pi \sum_{j=1}^m \sum_{k=1}^{m'} \frac{1 + \mu_f(j) + \overline{\mu_g(k)}}{2} \right) = \left( -\frac{1}{2\pi} \right)^\ell,
$$

$$
H_{0,2\ell}(x; r) = \frac{(2\pi)^\ell}{2\sqrt{\pi \ell}} \exp \left( -2\ell x^{1/\ell} \right) x^\theta \sum_{k=0}^r M_k x^{-k/\theta} + O \left( x^{\ell - r + 1/2\ell} \right),
$$

$$
\theta = \frac{1}{4\ell} \left( 1 - \ell + 2i \sum_{j=1}^m \sum_{k=1}^{m'} \Im(\mu_f(j)) - \Im(\mu_g(k)) \right) = \frac{1 - \ell}{4\ell},
$$

and the $M_k$’s are constants depending on $\ell$, $\mu_f(j)$ and $\mu_g(k)$. Here we have used that the summations over $\mu_f(j)$ and $\mu_g(k)$ vanish; that is

$$
\sum_{j=1}^m \mu_f(j) = 0 \quad \text{and} \quad \sum_{k=1}^{m'} \mu_g(k) = 0.
$$

Thus,

$$
\psi(x)G_{\tilde{f} \times g}(\pi^{2\ell} x^2 n^2) = \frac{(-1)^\ell}{2\sqrt{\pi \ell}} \sum_{\pm} \phi \left( \frac{x}{X} \right) e \left( \alpha x^\beta \pm \ell (xn)^{1/\ell} \right)
$$

$$
\times \sum_{k=0}^r M_k \left( \pi^{2\ell} x^2 n^2 e^{\mp i\pi \ell} \right)^{\ell - \frac{k}{2\ell}} + O \left( \phi \left( \frac{x}{X} \right) |(xn)^{2\theta - \frac{r + 1}{2\ell}} \right)
$$
where $\psi(x) = e(\alpha x^\beta) \phi \left( \frac{x}{X} \right)$. With this expansion the resonance sum in equation (2) becomes
\[
\sum_{n=1}^{\infty} \lambda_{f \times g}(n) \psi(n) = \frac{(-1)^{\ell}}{\sqrt{\ell}} \pi^{\ell^{-1}} \sum_{n=1}^{\infty} \sum_{\pm} \lambda_{f \times g}(n) 
\times \int_0^{\infty} \phi \left( \frac{x}{X} \right) e \left( \alpha x^\beta \pm \ell (xn)^{1/2} \right) \sum_{k=0}^{r} M_k \left( \pi^{2\ell} x^2 n^2 e^{\mp i\pi \ell} \right)^{\theta - \frac{1}{2\ell}} dx 
+ O \left( \sum_{n=1}^{\infty} |\lambda_{f \times g}(n)| \int_0^{\infty} \phi \left( \frac{x}{X} \right) (xn)^{2\theta - \frac{1}{2\ell}} dx \right).
\]
Doing the integral in the error term and making the change of variables $x \mapsto X^\ell$ we have
\[
\sum_{n=1}^{\infty} \lambda_{f \times g}(n) \psi(n) = \sum_{k=0}^{r} X^{1+2\theta - \frac{k}{r}} \sum_{n=1}^{\infty} \frac{\lambda_{f \times g}(n)}{n^{\frac{1}{r} - 2\theta}} \sum_{\pm} c_{k,\ell} I_k(n; \pm) 
+ O \left( X^{1+2\theta - \frac{r+1}{r}} \sum_{n=1}^{\infty} |\lambda_{f \times g}(n)| n^{2\theta - \frac{r+1}{r}} \right), \tag{10}
\]
where the constants have been condensed into $c_{k,\ell}$ and
\[
I_k(n; \pm) := \int_0^{\infty} \phi(t^\ell) e \left( \alpha X^{\beta} t^{\beta} \pm (Xn)^{1/2} \ell \right) t^{2\theta - k + \ell - 1} dt.
\]
Note that the error term here is $O_{\ell, r} \left( X^{1+2\theta - \frac{r+1}{r}} \right)$ for $r$ such that $L_{f \times g} \left( \frac{r+1}{\ell} - 2\theta \right)$ converges absolutely; i.e. for $r > \frac{\ell-1}{2}$.

The similarities between equation (10) above and equation (5.2) in [21] enable us to follow the argument given there. In particular, the integrals $I_k(n; \pm)$ are identical and so
\[
I_k(n; +) \ll \ell, j \ (nX)^{-\frac{j}{2}}
\]
for $j, n \geq 1$. We also need the bound
\[
\sum_{n=1}^{X} |\lambda_{f \times g}(n)| \ll X \tag{11}
\]
for $X > 0$ which follows by a Tauberian argument (Ch. XV Sec. 3 in [14]). The contribution of the terms in $\sum_+ \in$ equation (10) is
\[
\ll_{\ell, r} X^{1+2\theta - \frac{j}{2}} \sum_{n=1}^{\infty} \frac{|\lambda_{f \times g}(n)|}{n^{\frac{1}{r} - 2\theta}} \ll_{\ell, r} X^{1+2\theta - \frac{r+1}{r}}
\]
for \( j > r + 1 \). To estimate the terms in \( \sum \) let

\[
n_0 = \frac{1}{2} \min\{1, 2^{\beta - \frac{1}{2}}\} (\alpha \beta X^\beta)^\ell X^{-1} \quad \text{and} \quad n_1 = 2 \max\{1, 2^{\beta - \frac{1}{2}}\} (\alpha \beta X^\beta)^\ell X^{-1}.
\]

Then \( I_k(n; -) \ll (nX)^{-\frac{1}{2}} \) when \( n \notin (n_0, n_1) \) and the corresponding contribution is also \( O_{\ell, r} \left( X^{1+2\theta - \frac{r+1}{\ell}} \right) \). Equation (10) is thus reduced to

\[
\sum_{n=1}^{\infty} \lambda_{f \times g}(n) \psi(n) = \sum_{k=0}^{r} X^{1+2\theta - \frac{k}{\ell}} \sum_{n_0 < n < n_1} \frac{\lambda_{f \times g}(n)}{n^{\frac{\ell}{2} - 2\theta}} c_{k, \ell} I_k(n; -) + O_{\ell, r} \left( X^{1+2\theta - \frac{r+1}{\ell}} \right).
\]

(12)

Now either

\[
2 \max\{1, 2^{\beta - \frac{1}{2}}\} (\alpha \beta)^\ell \leq X^{1-\beta \ell} \quad \text{or} \quad 2 \max\{1, 2^{\beta - \frac{1}{2}}\} (\alpha \beta)^\ell > X^{1-\beta \ell}.
\]

In the former case \( n_1 \leq 1 \), the main term disappears, and we have

\[
\sum_{n=1}^{\infty} \lambda_{f \times g}(n) \psi(n) \ll_{\ell, \beta, r} X^{1+2\theta - \frac{r+1}{\ell}} \ll_{\ell, \beta, M} X^{-M}
\]

for any \( M > 0 \) by taking \( r \) sufficiently large in terms of \( M \). This proves part (i) of Theorem 2. In the latter case \( n_1 > 1 \) and there are two subcases: \( \beta \neq \frac{1}{\ell} \) or \( \beta = \frac{1}{\ell} \). If \( \beta \neq \frac{1}{\ell} \) then

\[
(\alpha X^\beta t^\beta - (nX)^{\frac{1}{2}\ell t})'' = \alpha (\ell \beta) (\ell \beta - 1) X^\beta t^{\beta-2} \gg_{\ell, \beta} \alpha X^\beta.
\]

Applying Lemma 5.1.3 of [8] we have \( I_k(n; -) \ll_{\beta, \ell} (\alpha X^\beta)^{-\frac{1}{2}} \), and using (11) the main term is

\[
\ll_{\ell, \beta} X^{1+2\theta} (\alpha X^\beta)^{-\frac{1}{2}} \sum_{n_0 < n < n_1} \frac{|\lambda_{f \times g}(n)|}{n^{\frac{\ell}{2} - 2\theta}} \ll_{\ell, \beta} (n_1 X)^{1+2\theta} (\alpha X^\beta)^{-\frac{1}{2}} \ll_{\ell, \beta} (\alpha X^\beta)^{\frac{1}{2}}.
\]

For the subcase \( \beta = \frac{1}{\ell} \) we take \( I_k(n; -) \ll 1 \) in equation (12) to obtain

\[
\ll_{\ell, \beta} X^{1+2\theta} \sum_{n_0 < n < n_1} \frac{|\lambda_{f \times g}(n)|}{n^{\frac{\ell}{2} - 2\theta}} \ll_{\ell, \beta} (n_1 X)^{1+2\theta} \ll_{\ell, \beta} (\alpha X^\beta)^{\frac{1}{2}}.
\]

This proves part (ii) of Theorem 2.

Note that when \( \beta = \frac{1}{\ell} \) we have \( I := (n_0, n_1) = \left( \frac{1}{2}, \frac{3}{2} \right) \times \left( \frac{\alpha \ell}{2 \theta} \right) \) and

\[
I_k(n; -) = \int_{0}^{\infty} \phi(t^\ell) e \left( (\alpha - \ell n^\frac{1}{2}) X^\frac{1}{2} t \right) t^{2\theta - k + \ell - 1} dt.
\]
Now \( n_1 > 1 \) so \( (\alpha/\ell)^k > 1/2 \) and hence there is a unique integer \( n_\alpha \geq 1 \) such that

\[
\left( \frac{\alpha}{\ell} \right)^k = n_\alpha + \lambda \quad \text{with} \quad -\frac{1}{2} < \lambda \leq \frac{1}{2}.
\]

Moreover, \( |n^{\frac{1}{\ell}} - \alpha/\ell| \gg \ell |n - n_\alpha|^{\alpha - \ell} \) for \( n \in I, \ n \neq n_\alpha \), and repeated integration by parts gives

\[
I_k(n; -) \ll_{\ell,j} \frac{1}{\left| n - n_\alpha \right|^{\alpha - \ell} X^{\frac{1}{\ell}}}
\]

for \( j \geq 0 \). Therefore, the contribution of the main terms in (12) without \( n = n_\alpha \) is

\[
\ll_{\ell} X^{1 + 2\theta} \left( \alpha^{\ell - 1} X^{-\frac{1}{\ell}} \right)^j \sum_{\substack{n_0 < n < n_1 \\ n \neq n_\alpha}} \frac{|\lambda f \times g(n)|}{n^{-2\theta}} \cdot \frac{1}{\left| n - n_\alpha \right|^j}
\]

\[
\ll_{\ell} X^{1 + 2\theta} \left( \alpha^{\ell - 1} X^{-\frac{1}{\ell}} \right)^j n_1^{\frac{k}{\ell}} \ll_{\ell} X^{1 + 2\theta} \left( \alpha^{\ell - 1} X^{-\frac{1}{\ell}} \right)^j \alpha^{\frac{k}{\ell}} \quad (13)
\]

for \( j \geq 1 \). Here we have used (11). Taking \( 0 < \varepsilon < \frac{1}{\ell} \) we have \( \alpha^{\ell - 1} X^{-\frac{1}{\ell}} < X^{-\varepsilon} \) whenever \( X > \alpha^{\frac{m(n-1)}{m+\varepsilon}} \). For \( j \) sufficiently large in terms of \( r \) this last expression is \( \ll X^{1 + 2\theta - \frac{r}{\ell}} \) in (13). Hence,

\[
\sum_{n=1}^{\infty} \lambda f \times g(n)\psi(n) = \frac{\lambda f \times g(n_\alpha)}{n_\alpha} \sum_{k=0}^{r} c_{l,r} I_k(n_\alpha; -)(n_\alpha X)^{1 + 2\theta - \frac{k}{\ell}} + O_{\ell,r,\varepsilon} \left( X^{1 + 2\theta - \frac{r-1}{\ell}} \right)
\]

which proves part \((iii)\) of Theorem 2. Finally, if \( \left( \frac{\alpha}{\ell} \right)^k = q \) is an integer, then \( \alpha = \ell q^{\frac{1}{\ell}} \) and \( n_\alpha = q \). Therefore,

\[
I_k(n_\alpha, -) = \int_0^{\infty} \phi(t^\ell) t^{2l\theta - k} dt = \int_0^{\infty} \phi(x) x^{\frac{1}{2\ell} - \frac{k}{2}} dx.
\]

This completes the proof of Theorem 2 with \( e(\alpha n^\beta) \); the proof for \( e(-\alpha n^\beta) \) is similar.

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