

Metrization Theorems

(Relates to text Sec. 34)

Introduction. What properties of a topological space (X, \mathcal{T}) are enough to guarantee that the topology actually is given by some metric? The space has to be normal, since we know metric spaces are normal. And the topology has to have a countable local basis at each point, since metric spaces have that property. In Chapter 6 of the text, there are theorems saying, “ (X, \mathcal{T}) is metric if and only if it has the following topological properties ...”. The conditions are (1) the space is regular, and (2) there is some countability property stronger than saying there is a countable local basis at each point, but a little weaker than 2nd-countable (but still strong enough that *regular + the property* \implies *normal*). In the current text section, the theorem is less general: we characterize *separable* metric spaces; but this is a good introduction to the ideas.

How does one prove that some topology on a space is given by a metric? There are two choices: either explicitly define the metric and prove the metric topology is the same as \mathcal{T} , or show that X is homeomorphic to a subspace of a known metric space. We used the first approach when we defined a metric on \mathbb{R}^ω that generates the product topology; and now we will see a good example of the other approach, which is also how the most general metrization theorems are proven.

Theorem (Urysohn metrization theorem). *If (X, \mathcal{T}) is a regular space with a countable basis for the topology, then X is homeomorphic to a subspace of the metric space \mathbb{R}^ω .*

The way I stated the above theorem, it is ambiguous: we have studied two (inequivalent) metrics for \mathbb{R}^ω : the product space metric and the uniform metric. The theorem is true with either metric, but it is an “if and only if” for the product metric. Recall that in the product topology, \mathbb{R}^ω has a countable dense subset: the set $S =$ all vectors (q_1, q_2, \dots) where each $q_i \in \mathbb{Q}$ and all but finitely many q_i are 0. Since \mathbb{R}^ω in the product topology is metrizable and has a countable dense subset, it must be 2nd-countable. Each subspace of space with a countable basis also has a countable basis. And, of course, each subspace of a metric space is metric. We conclude that each subspace of $(\mathbb{R}^\omega, \text{product metric})$ is metric and has a countable basis.

The above paragraph combines ideas from various parts of our course. So let us take this as one (bunch of) of our sample problems for the Final Exam. Your task is to organize the various facts, ideas, etc. into a coherent proof (and to be able to fill in details if asked); as always, an exam problem might involve filling in the details of one particular part of a longer argument).

Problem.*[sample for final exam]*

If (X, \mathcal{T}) is homeomorphic to a subspace of $(\mathbb{R}^\omega, \text{product topology})$, then (X, \mathcal{T}) is regular and has a countable basis.

Key steps:

- a. There is a metric on \mathbb{R}^ω that gives the product topology.
 - b. \mathbb{R}^ω in the product topology has a countable dense subset.
 - c. A metric space with a countable dense subset has a countable basis for the topology.
 - d. Each subspace of a 2nd-countable space is 2nd-countable.
 - e. Each subspace of a metric space is metrizable.
 - e. Each metric space is regular.
 - f. Put the pieces together.
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On the other hand, the metric space $(\mathbb{R}^\omega, \text{uniform metric})$ is not second-countable (so not separable) since it has an uncountable discrete subspace $K =$ the set of all vectors (t_1, t_2, \dots) where each $t_i = 0$ or 1 . The set of all sequences of 0's and 1's is uncountable, and the distance between any two elements of K is 1. So each subspace of $(\mathbb{R}^\omega, \text{uniform metric topology})$ is a metric space, but it need not be separable. We really should state the Urysohn metrization theorem as two theorems:

Theorem. (X, \mathcal{T}) is regular with a countable basis $\iff (X, \mathcal{T})$ is homeomorphic to a subspace of $(\mathbb{R}^\omega, \text{product topology metric})$.

Theorem. (X, \mathcal{T}) is regular with a countable basis $\implies (X, \mathcal{T})$ is homeomorphic to a subspace of $(\mathbb{R}^\omega, \text{uniform metric topology})$.

Proving the metrization theorem[s]. The text gives the details, so I will focus on the gestalt and some highlights. Our goal is to define an embedding of X into \mathbb{R}^ω . We want to assign to each point $x \in X$ a point $F(x) \in \mathbb{R}^\omega$, that is a (countably infinite) list of “coordinates”: $F(x) = (x_1, x_2, \dots)$. How can we find *numbers* that measure how a point $x \in X$ is related topologically to all the other points of X ? This is the bit of magic in this theorem. We will use Urysohn’s lemma infinitely many times to define a sequence of functions $f_n : X \rightarrow [0, 1]$; these will be the coordinate functions.

The space (X, \mathcal{T}) has a countable basis \mathcal{B} and it is regular, so it is normal. Given any closed set A and open neighborhood $U(A)$, there exists a Urysohn function for the disjoint closed sets $X - U$ and A . That is, there exists $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ for all $x \notin U$ and $f(a) = 1$ for all $a \in A$. In particular, for any pair B_n, B_m of elements of \mathcal{B} that happen to have $\bar{B}_n \subseteq B_m$, there exists a function $f : X \rightarrow [0, 1]$ with $f = 1$ on \bar{B}_n and $f = 0$ outside B_m .

Since \mathcal{B} is countable, the set of such pairs B_n, B_m is countable. Number these pairs in any order, and let f_1, f_2, \dots be the Urysohn functions defined in the preceding paragraph. Then define $F : X \rightarrow [0, 1]$ by

$$F(x) = (f_1(x), f_2(x), \dots).$$

We need to prove that the function F is 1-1, continuous, and has a continuous inverse (From $F(X) \rightarrow X$). The questions of continuity have to depend on what topology we use for \mathbb{R}^ω . But we can check injectivity before worrying about the topology.

Proposition. *The function $F : X \rightarrow \mathbb{R}^\omega$ is injective*

Proof. Suppose $x, y \in X$ with $x \neq y$. Since X is Hausdorff, there exist disjoint neighborhoods $U(x), V(y)$. Since \mathcal{B} is a basis, there exists some B_m with $x \in B_m \subseteq U$. Since X is regular, there exists a neighborhood $U'(x)$ such that $\bar{U}' \subseteq B_m$. And, again since \mathcal{B} is a basis, there exists a basis set B_n with $x \in B_n \subseteq U'$. Since $\bar{U}' \subseteq B_m$, we thus have $\bar{B}_n \subseteq B_m$. The Urysohn function f_j associated to the pair B_n, B_m has $x \rightarrow 1$ and $y \rightarrow 0$; so $F(x) \neq F(y)$. \square

The text goes on to show that, in the product topology, F is continuous and has a continuous inverse. The proof that F is continuous is easy because each coordinate function is continuous; the proof that F is an open map takes more work; see the text for the details.

To use the uniform topology, we need to change F . Recall that in the product topology, if we are studying a function from a space into a product space, i.e. some $G : Y \rightarrow \prod_{\alpha \in J} X_\alpha$, and we want to show that G is continuous, it is sufficient to check that each component function $G_\alpha : Y \rightarrow X_\alpha$ is continuous. But in the uniform topology, this is not sufficient.

Example (Page 127, problem 4a). *The function $G : \mathbb{R} \rightarrow \mathbb{R}^\omega$ defined by $G(t) = (t, 2t, 3t, 4t, \dots)$ is not continuous in the uniform topology on \mathbb{R}^ω . In particular, there is no neighborhood of 0 that is mapped by G into a uniform ϵ -neighborhood of $(0, 0, 0, \dots)$.*

To make the coordinate function F topologically “well-behaved” for the uniform metric on \mathbb{R}^ω , we need to eliminate the difficulty suggested by the above example. We do this by making the coordinate functions f_j get smaller as j gets larger. Specifically, define

$$G : X \rightarrow \mathbb{R}^\omega \text{ by } G(x) = (f_1(x), \frac{1}{2}f_2(x), \frac{1}{3}f_3(x), \frac{1}{4}f_4(x), \dots)$$

There is one more part of the proofs that is “cute”, “clever”, or ‘annoyingly slick’, depending on your tastes: The uniform metric topology is finer than the product topology on \mathbb{R}^ω . SO once we know F is an open map in the product topology (that takes work), it is easy to see that F , hence G , is an open map in the uniform topology. Conversely, once we know G is continuous in the uniform topology (that takes work), it is easy to see that F is continuous in the product topology.

Here are some sample problems for the Final Exam that you can use to solidify your understanding of these proofs. The first is an easier special case; the others are the “standard” Urysohn metrization theorem.

Problem.

[sample for final exam]

Prove: If X is a compact Hausdorff space with a countable basis, then there exists an embedding of X into \mathbb{R}^ω , where \mathbb{R}^ω has the product topology.

Problem.

[sample for final exam]

Write a one-to-two page proof:

If X is a regular space with a countable basis, then there exists an embedding of X into \mathbb{R}^ω , where \mathbb{R}^ω has the uniform topology.

Problem.

[sample for final exam]

Write a one-to-two page proof:

If X is a regular space with a countable basis, then there exists an embedding of X into \mathbb{R}^ω , where \mathbb{R}^ω has the uniform topology.

(end of handout)