A FLIP-BOOK OF WAVELETS ON THE INTERVAL $[0,5]$ AND THEIR ASSOCIATED SCALING FUNCTIONS

In Chapters 1–2 in [BrJo02] and in [Tre01b], it is pointed out that families of compactly supported wavelets admit a group-theoretic formulation. When this idea is specialized to the case of multiresolution wavelets which have both the scaling function (father function φ) and the wavelet generator (mother function ψ) itself supported in the fixed interval from 0 to 5, then the full variety of possibilities may be described by two independently varying unit vectors in $\mathbb{C}^2$, where the unitarity of $v = (v_1, v_2) \in \mathbb{C}^2$ refers to $|v_1|^2 + |v_2|^2 = 1$. Unit vectors in $\mathbb{C}^2$ define pure quantum-mechanical states. The latter may be parameterized by points on the (Bloch) sphere S^2. For example $(\cos\theta, \sin\theta)$ in $\mathbb{C}^2$ corresponds to the point $(\sin 2\theta, 0, \cos 2\theta)$ on S^2. Hence viewing S^2 as embedded in $\mathbb{R}^3$, the vector moves on a great circle on S^2 in the (x_1, x_3)-plane. This is reflected in the fact that formulas (1.2.3)–(1.2.4) in [BrJo02] for the masking coefficients $a_k(\theta, \rho)$ are in fact functions of 2θ and 2ρ. In order to obtain a graphic illustration of the various shapes and forms of the two functions φ (on the right) and ψ (on the left), we restrict attention here to two vectors of the form $(\cos\theta, \sin\theta)$, $(\cos\rho, \sin\rho)$ with the parameters θ and ρ varying independently. These are in fact the vectors that give rise to real-valued functions φ and ψ. The functions φ and ψ are generated by a subdivision algorithm which is described in Chapter 1 of [BrJo02]; and the subdivision in turn is determined by a set of masking coefficients $a_k(\theta, \rho)$, $k = 0, 1, 2, 3, 4, 5$. Hence the two functions φ and ψ depend on variations in (θ, ρ); and this is illustrated graphically in the following supplement to [BrJo02]. It may be used as a flip-book of wavelets, or it may be explored with a mouse-search on the screen. While the functions φ and ψ are square-integrable, they are continuous only for (θ, ρ) in a subset of $[0, \pi) \times [0, \pi)$, with points outside that square identified with points inside it through periodicity. (This subset is often identified by vanishing moments or by spectral conditions; see [CGV99].) Hence continuous dependence on (θ, ρ) must be measured in mean square, i.e., in the metric defined by the L^2-norm of functions on $[0, 5]$. This continuity follows from [BrJo02, Theorem 2.5.8], and receives a graphic illustration in what follows. Actually there is a finite number of exceptions to the continuity. They occur when the wavelet is one of the degenerate Haar cases. These are the Haar wavelets which do not define strict orthonormal bases, but only tight frames. They have φ of the form $\varphi = c\chi_I$ where I is a subinterval of $[0, 5]$ of length 3 or 5, and where $c = \frac{1}{3}$ in the first case, and $c = \frac{1}{5}$ in the second. If the reader follows the moving wavelet pictures (on a printout, or on the screen), we hope he/she will get some intuitive ideas of fundamental wavelet relationships. The rigorous mathematics relating the various continuity properties to cascade approximation, to moments, and to spectral estimates, is covered in much more detail in the references below, especially [BrJo02] and [CGV99].

References

[BrJo02] O. Bratteli and P.E.T. Jorgensen, *Wavelets through a Looking Glass: The World of the Spectrum*, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2002.

[CGV99] A. Cohen, K. Gröchenig, and L.F. Villemoes, *Regularity of multivariate refinable functions*, Constr. Approx. **15** (1999), 241–255.

[Jor01b] P.E.T. Jorgensen, *Minimality of the data in wavelet filters*, Adv. Math. **159** (2001), 143–228.

[Tre01b] B.F. Treadway, Appendix to [Jor01b].

Notes on computation of cascade approximants

Once the masking coefficients $a_0, a_1, \ldots, a_5$ are specified as functions of the parameters θ and ρ, the computation of cascade approximants of the scaling function φ is done with the following Mathematica operations. (The Mathematica variables a0, a1, etc., are normalized to include a factor of $\sqrt{2}$ that appears in the cascade iteration.)

```
loctwont[θ_, ρ_] :=
 N[Transpose[{{a4[θ, ρ], a2[θ, ρ], a0[θ, ρ]}, {a5[θ, ρ], a3[θ, ρ], a1[θ, ρ]}}]]

cascadestep[phitable_, θ_, ρ_] :=
 Flatten[Partition[Flatten[{0, 0, phitable, 0, 0}], 3, 1].loctwont[θ, ρ]]

wavelet[phistart_, itercount_, θ_, ρ_] :=
 Transpose[{Table[i (2^(-itercount)), {i, 0, 5 (2^itercount) - 5}],
   Nest[cascadestep[#, θ, ρ] &, phistart, itercount]}]
```

Each cascade step works with the list of values from the previous step (phitable), pads it with zeroes on left and right, works it up into a matrix by overlapping divisions (the Partition operation), does a matrix multiplication with a matrix of masking coefficients (loctwont), and reduces (with Flatten) the resulting matrix to a list again. At each stage the implicit grid spacing is halved, and at the specified final iteration, the values of these grid points are associated with the elements of the list (using Table and Transpose), so that a plot of the scaling-function approximant can be made with other Mathematica operations such as ListPlot.

A direct implementation in Mathematica of the computation of ψ from φ by the formula [BrJo02, (2.5.25)],

$$\psi(x) = \sqrt{2} \sum_{k=0}^{5} (-1)^k a_{5-k} \varphi(2x - k),$$

is (again with $\sqrt{2}$ subsumed in a0, a1, ...)

```
embedwavelet[phistart_, itercount_, θ_, ρ_] :=
 Flatten[{Table[0, {i, -(5 (2^itercount) - 5), -1}], Nest[cascadestep[#, θ, ρ] &, phistart,
    itercount], Table[0, {i, (5 (2^itercount) - 5) + 1, 2 (5 (2^itercount) - 5)}]}, 1]

avec[θ_, ρ_] := {a0[θ, ρ], a1[θ, ρ], a2[θ, ρ], a3[θ, ρ], a4[θ, ρ], a5[θ, ρ]}

waveletpsi[phistart_, itercount_, θ_, ρ_] :=
 Transpose[{Table[i (2^(-itercount - 1)), {i, 0, 2 (5 (2^itercount) - 5)}],
   Table[Sum[((-1)^(5 - k) avec[θ, ρ][[k + 1]] embedwavelet[phistart, itercount, θ,
        ρ][[j + k (2^itercount - 1)]]), {k, 0, 5}], {j, 1, 2 (5 (2^itercount) - 5) + 1}]}]
```

But using the Table and Sum operations in this way is inefficient for two reasons. First, the sum has, after suitable rearrangement, the form of a sequence correlation that can be implemented with a fast Fourier transform. And second, the way the operations above are interpreted, the same scaling function (embedwavelet) is computed repeatedly for each term of the sum, rather than being computed once and saved.

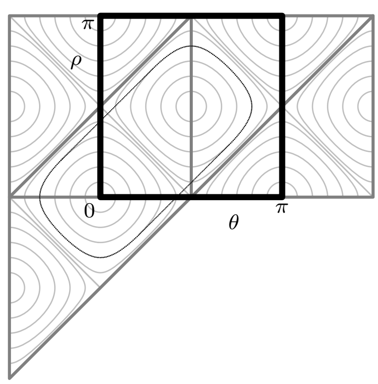
Both of these inefficiencies can be remedied by the use of the ListCorrelate operation.

```
correlatewavelet[phistart_, itercount_, θ_, ρ_] :=
 Flatten[Transpose[Partition[PadLeft[PadRight[Nest[cascadestep[#, θ, ρ] &, phistart,
      itercount], 6 (2^itercount) - 6], 11 (2^itercount) - 11], (2^itercount) - 1]]]

signavec[θ_, ρ_] := {-a0[θ, ρ], a1[θ, ρ], -a2[θ, ρ], a3[θ, ρ], -a4[θ, ρ], a5[θ, ρ]}

correlatewaveletpsi[phistart_, itercount_, θ_, ρ_] :=
 Transpose[{Table[i (2^(-itercount - 1)), {i, 0, 11 (2^itercount) - 12}],
   Flatten[Transpose[Partition[ListCorrelate[signavec[θ, ρ],
      correlatewavelet[phistart, itercount, θ, ρ], {1, 1}, 0], 11]]]}]
```

The necessary rearrangement amounts to grouping the list of values into a matrix and transposing the matrix. This is done, at both ends of the computation, with the Partition and Transpose operations, followed by Flatten to return to an ungrouped list. Note also the use of PadLeft and PadRight to add zeroes to keep the different rows of the matrix from getting mixed in the ListCorrelate operation; these could have been used in the direct implementation as well, where zero-padding was needed at the ends of the sum.

This Fourier-transform method gives numerically identical results, thanks in part to Mathematica's implementation that uses a real transform method on real data. For the wavelet functions computed for this flip-book, the method using ListCorrelate works about 500 times faster than the method using Table and Sum.

The guide diagrams in the middle column show the (θ, ρ) plane with dots for the point (and its periodic replicas) whose wavelet function and scaling function are displayed on the current page and line. The square outlined in black is the fundamental region $[0, \pi) \times [0, \pi)$, shown with labels below.



The first part of the path, shown in the top line, follows the vanishing-moment curve, where the wavelet function $\psi(x)$ satisfies the moment condition $\int x\,\psi(x)\,dx = 0$, and the masking coefficients $a_0, \ldots, a_5$ satisfy the equivalent divisibility condition $m_0(z) = 2^{-\frac{1}{2}} \sum_{k=0}^{5} a_k z^k = (z+1)^2\, p(z)$ for some polynomial p. The viewer familiar with [Jor01b] may find amusement in watching for the appearance in the sequence of the familiar Daubechies wavelets, as well as the single point in the sequence where the additional moment and divisibility conditions $\int x^2\,\psi(x)\,dx = 0$ and $m_0(z) = (z+1)^3\, q(z)$ are met.