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Iterated function systems and permutation representations of the Cuntz algebra. (English. English summary)

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Isometries, shifts, Cuntz algebras and multiresolution wavelet analysis of scale N. (English. English summary)

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FEATURED REVIEW.

The main theme of these two articles is the study of some representations of the Cuntz algebra $\mathcal{O}_N$, coming from suitable dynamical systems in the first one and from wavelets in the second one. These studies constitute a generalization and an improvement of work by the authors and G. L. Price [in Quantization, nonlinear partial differential equations, and operator algebra (Cambridge, MA, 1994), 93–138, Proc. Sympos. Pure Math., 59, Amer. Math. Soc., Providence, RI, 1996; MR 97h:46107] and Jorgensen and S. Pedersen [Constr. Approx. 12 (1996), no. 1, 1–30; MR 97c:46091], among others.

Before discussing the content of the articles, let us recall that $\mathcal{O}_N$ is the C^*-algebra generated by N isometries $s_0, s_1, \cdots, s_{N-1}$ satisfying (1) $s_i^* s_j = \delta_{ij}$ and (2) $\sum_{i=0}^{N-1} s_i s_i^* = 1$. It is a simple C^*-algebra, and every system of operators $\{S_0, S_1, \cdots, S_{N-1}\}$ on a Hilbert space $\mathcal{H}$ satisfying relations (1) and (2) determines a representation of $\mathcal{O}_N$.

The study of the representations of $\mathcal{O}_N$ on $\mathcal{H}$ is not only interesting in itself, but also provides endomorphisms of $B(\mathcal{H})$: If $S_0, \cdots, S_{N-1} \in B(\mathcal{H})$ satisfy (1) and (2), then the map $\alpha\colon A \mapsto \sum_i S_i A S_i^*$ is an endomorphism of $B(\mathcal{H})$, and, conversely, every endomorphism is of this form. Moreover, such an endomorphism is a shift in the sense of Powers if and only if the corresponding representation is irreducible when restricted to the subalgebra $\mathrm{UHF}_N = \{a \in \mathcal{O}_N;\ \gamma_z(a) = a$ for all $z \in \mathbf{T}\}$, where γ_z is the automorphism of $\mathcal{O}_N$ defined by $\gamma_z(s_i) = z s_i$.

All representations dealt with are particular cases of the following scheme: $\mathcal{H} = L^2(\Omega, \mu)$, where Ω is a measure space and μ is a probability measure on Ω. Assume that there are N maps $\sigma_i\colon \Omega \to \Omega$ with the property that (3) $\mu(\sigma_i(\Omega) \cap \sigma_j(\Omega)) = 0$ for $i \neq j$, (4) $\mu(\sigma_i(\Omega)) = 1/N$, so that $\{\sigma_0(\Omega), \cdots, \sigma_{N-1}(\Omega)\}$ is a partition of Ω up to measure zero. Assume furthermore that (5) $\mu(\sigma_i(Y)) = 1/N$ for every measurable subset Y of Ω. Then the σ_i's are injections up to measure zero, and hence it is possible to define an N-to-1 map $\sigma\colon\Omega\to\Omega$ (well defined up to measure zero) by $\sigma\circ\sigma_i=\sigma_i$ for $i\in\mathbf{Z}_N=\{0,\cdots,N-1\}$. Finally, the announced representations $s_i\mapsto S_i$ of $\mathcal{O}_N$ on $L^2(\Omega,\mu)$ are defined by using N measurable functions $m_0,\cdots,m_{N-1}\colon\Omega\to\Omega$ with the property that the $N\times N$ matrix (6) $N^{-1/2}(m_i(\sigma_j(x)))_{0\le i,j\le N-1}$ is unitary for almost all $x\in\Omega$. Then, setting (7) $(S_i\xi)(x)=m_i(x)\xi(\sigma(x))$, one gets a representation π of $\mathcal{O}_N$.

For instance, take $\Omega=\mathbf{T}$ with its normalized Haar measure, and set $\sigma_k(e^{2\pi i\theta})=\exp(2\pi i(\theta+k)/N)$, so that $\sigma(z)=z^N$. Moreover, choose integers $r_0,\cdots,r_{N-1}$ that are pairwise incongruent mod N and define (8) $(S_k\xi)(z)=z^{r_k}\xi(z^N)$, for $\xi\in L^2(\mathbf{T})$. With respect to the natural basis of $L^2(\mathbf{T})$, the corresponding representation of $\mathcal{O}_N$ is permutative in the following general sense: There exists an orthonormal basis $(e_n)_{n\in\mathbf{N}}$ of $\mathcal{H}$ such that (9) $S_ke_n\in\{e_m;\ m\in\mathbf{N}\}$.

The first article under review is mainly devoted to the study of general permutative representations of $\mathcal{O}_N$; it contains the construction of a universal permutative (nonseparable) representation, a detailed analysis of the case $N=2$ based on arithmetic and combinatorial properties of $\mathbf{Z}$ and other classes of representations associated to pairs $(\mathbf{N},D)$ where $\mathbf{N}$ is a suitable integer $(\nu\times\nu)$-matrix and where $D\subset\mathbf{Z}^\nu$ plays the role of the r_k's in the above; the associated representation acts on $L^2(\mathbf{T}^\nu)$ and is defined as in (8). Their study requires a self-similar compact subset $\Omega\in\mathbf{R}^\nu$, and several examples are treated.

We now review the second article. The authors start by studying isometries on $L^2(\mathbf{T})$ of the form (10) $(S_m\xi)(z)=m(z)\xi(z^N)$. (Such an operator is an isometry if and only if $N^{-1}\sum_{w:w^N=z}|m(w)|^2=1$.) Using the so-called Wold decomposition (into a unitary part and a "shift" part), they prove that the unitary part of S_m is one- or zero-dimensional. Moreover, it is one-dimensional if and only if $|m(z)|=1$ a.e. and there exist a measurable $\xi\colon\mathbf{T}\to\mathbf{T}$ and $\lambda\in\mathbf{T}$ such that $m(z)\xi(z^N)=\lambda\xi(z)$ a.e. Furthermore, they are able to characterize representations of $\mathcal{O}_N$ generated by isometries S_{m_i} as in (10), where $m_i=\sqrt{N}\chi_{A_i}u$ with suitable measurable $A_0,\cdots,A_{N-1}\subset\mathbf{T}$ and $u\colon\mathbf{T}\to\mathbf{T}$: these are representations π^u on $L^2(\mathbf{T})$ for which the elements of $\pi^u(\mathcal{D}_N)''$ are multiplication operators by functions in $L^\infty(\mathbf{T})$ (recall that $\mathcal{D}_N$ is the closed linear span of $\{s_Is_I^*;\ I$ a multi-index set$\}$). Moreover, for specific A_i's, they classify these representations: π^u is equivalent to $\pi^{u'}$ if and only if there exists a measurable function $\Delta\colon\mathbf{T}\to\mathbf{T}$ such that (11) $\Delta(z)u(z)=u'(z)\Delta(z^N)$ a.e.

Finally, a connection is made between the above representations and wavelets; for instance, it is proved that if S_{m_0} (as in (10)) is a shift in the sense that $\bigcap_{n\ge1}S_{m_0}^nL^2(\mathbf{T})=0$, then S_{m_0} is a compression

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of the scaling operator $(U_N\xi)(x) = N^{-1/2}\xi(x/N)$ which acts on $L^2(\mathbf{R})$ and which appears in wavelet analysis of scale N. Unfortunately, it is impossible to give more details on that interesting construction here because in order to do so we would have to rewrite some parts of the article. *Paul Jolissaint* (CH-NCH)