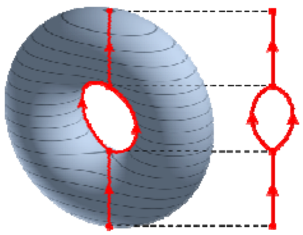
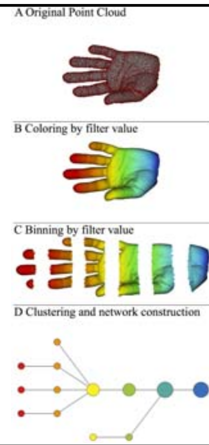
https://en.wikipedia.org/wiki/Reeb_graph





http://www.nature.com/srep/2013/130207/srep01236/full/srep01236.html

mapper.filters.kNN_distance(data, k, metricpar={}, callback=None)

The distance to the k-th nearest neighbor as an (inverse) measure of density.

Note how the number of nearest neighbors is understood:

k=1, the first neighbor, makes no sense for a filter function since the first nearest neighbor of a data point is always the point itself, and hence this filter function is constantly zero.

The parameter k=2 measures the distance from x_i to the nearest data point other than x_i itself.

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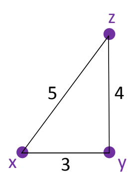
$\delta_1(x) =$

$\delta_2(x) =$

$\delta_3(x) =$

$\delta_4(x) =$



If x is in a denser region than y, then $\delta_k(x)$ $\delta_k(y)$

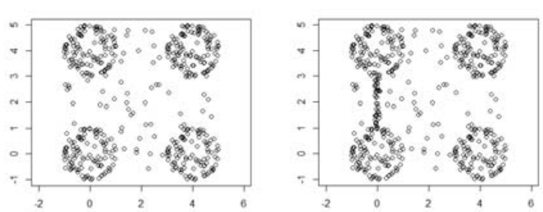
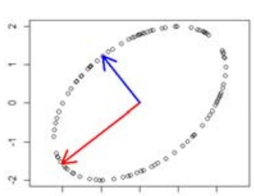
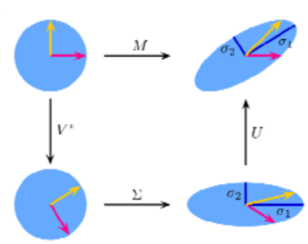


Image of a circle under linear transformation T(x) = Ax where A is a symmetric matrix



https://en.wikipedia.org/wiki/File:Singular-Value-Decomposition.svg



$$M = U \cdot \Sigma \cdot V^*$$

The SVD decomposes *M* into three simple transformations:

- a rotation *V**,
- a scaling Σ along the coordinate axes and
- a second rotation *U*.

5.1: Eigenvalues and Eigenvectors

Defn: λ is an **eigenvalue** of the matrix A if there exists a <u>nonzero</u> vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

The vector $\mathbf{x}$ is said to be an **eigenvector** corresponding to the eigenvalue λ.

Example: Let $A = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of A and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ for any k.

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is <u>NOT</u> an eigenvector of A.

MOTIVATION:

Note $\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus $A\begin{bmatrix} 2 \\ 8 \end{bmatrix} = A(\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = A\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3A\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= -1\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \cdot 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$

Finding eigenvalues:

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ (Note A is a SQUARE matrix).

Then $A\mathbf{x} = \lambda I\mathbf{x}$ where I is the identity matrix.

Thus $A\mathbf{x} - \lambda I\mathbf{x} = (A - \lambda I)\mathbf{x} = \mathbf{0}$

Thus if $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Thus $det(A - \lambda I)\mathbf{x} = 0$.

Note that the eigenvectors corresponding to λ are the nonzero solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Thus to find the eigenvalues of A and their corresponding eigenvectors:

Step 1: Find eigenvalues: Solve the equation

$$det(A - \lambda I) = 0 \text{ for } \lambda.$$

Step 2: For each eigenvalue λ_0, find its corresponding eigenvectors by solving the homogeneous system of equations

$$(A - \lambda_0 I)\mathbf{x} = 0 \text{ for } \mathbf{x}.$$

Defn: $det(A - \lambda I) = 0$ is the **characteristic equation** of A.

Defn: The **eigenspace** corresponding to an eigenvalue λ_0 of a matrix A is the set of all solutions of $(A - \lambda_0 I)\mathbf{x} = \mathbf{0}$.

Note: An eigenspace is a vector space

The vector $\mathbf{0}$ is always in the eigenspace.

The vector $\mathbf{0}$ is never an eigenvector.

The number 0 can be an eigenvalue.

Thm: A square matrix is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Defn: A matrix is **symmetric** if $A = A^T$.

Recall An invertible matrix P is **orthogonal** if $P^{-1} = P^T$

Defn: A matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix P such that $P^{-1}AP = D$ where

Defn: Suppose the characteristic polynomial of A is

$$(\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2}...(\lambda - \lambda_n)^{k_p}$$

where the λ_i, $i = 1, ..., p$ are DISTINCT. Then the **algebraic multiplicity of** λ_i is k_i.

That is the **algebraic multiplicity of** λ_i is the number of times that $(\lambda - \lambda_i)$ appears as a factor of the characteristic polynomial of A.

Defn: The **geometric multiplicity of** λ_i
= dimension of the eigenspace corresponding to λ_i.

Thm (Geometric and Algebraic Multiplicity): The geometric multiplicity is less than or equal to the algebraic multiplicity [That is, Nullity of $(A - \lambda_i I) \leq k_i$].

Thm: The following are equivalent for an $n \times n$ matrix A:

a.) A is orthogonally diagonalizable.

b.) There exists an orthonormal basis for R^n consisting of eigenvectors of A.

c.) A is symmetric.

Thm: If A is a symmetric matrix, then:

a.) The eigenvalues of A are all real numbers.

b.) Eigenvectors from different eigenspaces are orthogonal.

c.) Geometric multiplicity of an eigenvalue = its algebraic multiplicity

Suppose $A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $A \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

Claim:

If $A = A^T$ and $\lambda_1 \neq \lambda_2$, then $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is perpendicular to $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

I.e., If eigenvectors come from different eigenspaces, then the eigenvectors are orthogonal WHEN $A = A^T$.

Pf of claim: $\lambda_1(v_1, v_2) \cdot (w_1, w_2) = \lambda_1[v_1, v_2] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$$= (\lambda_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix})^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = (A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix})^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T A^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = [v_1, v_2] A \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= [v_1, v_2]\lambda_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda_2 [v_1, v_2] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \lambda_2(v_1, v_2) \cdot (w_1, w_2)$$

$\lambda_1(v_1, v_2) \cdot (w_1, w_2) = \lambda_2(v_1, v_2) \cdot (w_1, w_2)$

implies $\lambda_1(v_1, v_2) \cdot (w_1, w_2) - \lambda_2(v_1, v_2) \cdot (w_1, w_2) = 0$.

Thus $(\lambda_1 - \lambda_2)(v_1, v_2) \cdot (w_1, w_2) = 0$

$\lambda_1 \neq \lambda_2$ implies $(v_1, v_2) \cdot (w_1, w_2) = 0$

Thus these eigenvectors are orthogonal.

IF A is symmetric,

To orthogonally diagonalize a symmetric matrix A:

1.) Find the eigenvalues of A.

Solve $det(A - \lambda I) = 0$ for λ.

2.) Find a basis for each of the eigenspaces.

Solve $(A - \lambda_j I)\mathbf{x} = 0$ for $\mathbf{x}$.

3.) Use the Gram-Schmidt process to find an orthonormal basis for each eigenspace.

That is for each λ_j use Gram-Schmidt to find an orthonormal basis for $Nul(A - \lambda_j I)$.

Eigenvectors from different eigenspaces will be orthogonal, so you don't need to apply Gram-Schmidt to eigenvectors from different eigenspaces

4.) Use the eigenvalues of A to construct the diagonal matrix D, and use the orthonormal basis of the corresponding eigenspaces for the corresponding columns of P.

5.) Note $P^{-1} = P^T$ since the columns of P are orthonormal.

Example 1:

Orthogonally diagonalize $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Step 1: Find the eigenvalues of A:

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4$$

$$= \lambda^2 - 5\lambda + 4 - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$$

Thus $\lambda = 0, 5$ are are eigenvalues of A.

2.) Find a basis for each of the eigenspaces:

$$\lambda = 0 : (A - 0I) = A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Thus $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 0.

$$\lambda = 0 : (A - 5I) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

Thus $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A with eigenvalue 5.

3.) Create orthonormal basis:

Since A is symmetric and the eigenvectors $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ come from different eigenspaces (ie their eigenvalues are different), these eigenvectors are orthogonal. Thus we only need to normalize them:

$$\left|\left| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right|\right| = \sqrt{4+1} = \sqrt{5}$$

$$\left|\left| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right|\right| = \sqrt{1+4} = \sqrt{5}$$

Thus an orthonormal basis for $col(A) = R^2 = \left\{ \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \right\}$

4.) Construct D and P

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Make sure order of eigenvectors in D match order of eigenvalues in P.

5.) P orthonormal implies $P^{-1} = P^T$

Thus $P^{-1} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

Note that in this example, $P^{-1} = P$, but that is NOT normally the case.

Thus $A = PDP^{-1}$

Thus $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

Example 2:
Orthogonally diagonalize $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Step 1: Find the eigenvalues of A:

$$det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 0 & -\lambda & -\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -\lambda & -\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ -\lambda & -\lambda \end{vmatrix} + 0\begin{vmatrix} -1 & 1 \\ 1-\lambda & -1 \end{vmatrix}$$

$= (1-\lambda)[(1-\lambda)(-\lambda) - \lambda] + [\lambda + \lambda]$
$= (1-\lambda)(-\lambda)[(1-\lambda)+1] + 2\lambda = (1-\lambda)(-\lambda)(2-\lambda) + 2\lambda$
Note I can factor out $-\lambda$, leaving only a quadratic to factor:
$= -\lambda[(1-\lambda)(2-\lambda) - 2]$
$= -\lambda[\lambda^2 - 3\lambda + 2 - 2] = -\lambda[\lambda^2 - 3\lambda] = -\lambda^2[\lambda - 3]$

Thus their are 2 eigenvalues:
$\lambda = 0$ with algebraic multiplicity 2. Since A is symmetric, geometric multiplicity = algebraic multiplicity = 2. Thus the dimension of the eigenspace corresponding to $\lambda = 0$
$[= Nul(A - 0I) = Nul(A)]$ is 2.
$\lambda = 3$ w/algebraic multiplicity = 1 = geometric multiplicity.

Thus we can find an orthogonal basis for R^3 where two of the basis vectors comes from the eigenspace corresponding to eigenvalue 0 while the third comes from the eigenspace corresponding to eigenvalue 3.

2.) Find a basis for each of the eigenspaces:

2a.) $\lambda = 0 : A - 0I = A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus a basis for eigenspace corresponding to eigenvalue 0 is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

We can now use Gram-Schmidt to turn this basis into an orthogonal basis for the eigenspace corresponding to eigenvalue 0 or we can continue finding eigenvalues.

3a.) Create orthonormal basis using Gram-Schmidt for the eigenspace corresponding to eigenvalue 0:

Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$proj_{\mathbf{v_1}}\mathbf{v_2} = \frac{\mathbf{v_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1} = \frac{-1+0+0}{1+1+0}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

The vector component of $\mathbf{v_2}$ orthogonal to $\mathbf{v_1}$ is

$$\mathbf{v_2} - proj_{\mathbf{v_1}}\mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Thus an orthogonal basis for the eigenspace corresponding to eigenvalue 0 is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

To create orthonormal basis, divide each vector by its length:

$$\left\| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

$$\left\| \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\| = \sqrt{\tfrac{1}{4} + \tfrac{1}{4} + 1} = \sqrt{\tfrac{3}{2}}$$

Thus an orthonormal basis for the eigenspace corresponding to eigenvalue 0 is

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2\sqrt{3}} \\ \frac{\sqrt{2}}{2\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \right\}$$

2b.) Find a basis for eigenspace corresponding to $\lambda = 3$:

$$A - 3I = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for eigenspace corresponding to eigenvalue 3 is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

FYI: Alternate method to find 3rd vector: Since you have two linearly independent vectors from the eigenspace corresponding to eigenvalue 0, you only need one more vector which is orthogonal to these two to form a basis for R^3. Note since A is symmetric, any such vector will be an eigenvector of A with eigenvalue 3. Note this shortcut only works because we know what the eigenspace corresponding to eigenvalue 3 looks like: a line perpendicular to the plane representing the eigenspace corresponding to eigenvalue 0.

3b.) Create orthonormal basis for the eigenspace corresponding to eigenvalue 3:

We only need to normalize:

$$\left\| \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

Thus orthonormal basis for eigenspace corresponding to eigenvalue 3 is

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\}$$

4.) Construct D and P

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Make sure order of eigenvectors in D match order of eigenvalues in P.

5.) P orthonormal implies $P^{-1} = P^T$

Thus $P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Thus $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = A = PDP^{-1} =$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$