Let $C_{p-1} =< e_1, ..., e_k >$ and $C_p =< f_1, ..., f_m >$.

Then $C^{p-1} =< \epsilon_1, ..., \epsilon_k >$ and $C^p =< \phi_1, ..., \phi_m >$

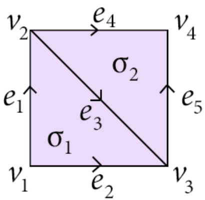
where $\epsilon_i : C_{p-1} \to \mathbb{Z}_2, \quad \epsilon_i(e_j) = \begin{cases} 1 & \textit{if} \quad i = j \\ 0 & \textit{otherwise} \end{cases}$

and $\phi_i : C_{p-1} \to \mathbb{Z}_2, \quad \phi_i(f_j) = \begin{cases} 1 & \textit{if} \quad i = j \\ 0 & \textit{otherwise} \end{cases}$

And $\partial_p : C_p \to C_{p-1}$ and $\delta_{p-1} : C^{p-1} \to C^p$ are defined by matrix $M = (m_{ij})$ where

$$m_{ij} = \begin{cases} 1 & \textit{if} \;\; e_j \in \partial(f_i) \\ 0 & \textit{otherwise} \end{cases} \quad \text{and} \quad m_{ji} = \begin{cases} 1 & \textit{if} \;\; e_j \in \partial(f_i) \\ 0 & \textit{otherwise} \end{cases}$$

I.e., $\partial_p = M$ and $\delta_{p-1} = M^T$.



$$\partial_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \delta_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\delta(\epsilon_1)(\sigma_1) = \epsilon_3(\partial(\sigma_1)) = \epsilon_3(e_1 + e_2 + e_3) = 1 + 0 + 0 = 1$$

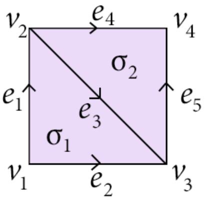
$$\delta(\epsilon_1)(\sigma_2) = \epsilon_3(\partial(\sigma_2)) = \epsilon_3(e_3 + e_4 + e_5) = 0. \text{ Thus } \delta(\epsilon_1) = \phi_1$$

$$\delta(\epsilon_3)(\sigma_1) = \epsilon_3(\partial(\sigma_1)) = \epsilon_3(e_1 + e_2 + e_3) = 0 + 0 + 1 = 1$$

$$\delta(\epsilon_3)(\sigma_2) = \epsilon_3(\partial(\sigma_2)) = \epsilon_3(e_3 + e_4 + e_5) = 1.$$

Thus $\delta(\epsilon_1) = \phi_1 + \phi_2$

https://jeremykun.com/tag/simplicial-complex/



$$Z_p = nul(\partial_p),\ B_p = im(\partial_{p+1})$$

$$Z^p = nul(\delta_p),\ B^p = im(\delta^{p-1})$$

$$\partial_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \delta_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Thus $rank(B_{p-1}) = rank(\partial_p) = rank(\delta_{p-1}) = rank(B^p)$

$$rank(Z_p) + rank(B_{p-1}) = rank(C_p) = rank(C^p) = rank(Z^p) + rank(B^{p+1})$$

Hence $rank(Z_p) + rank(B^p) = rank(Z^p) + rank(B_p)$

$$rank(H_p) = rank(Z_p) - rank(B^p) = rank(Z^p) + rank(B^p) = rank(H^p)$$

Given filtration $K_1 \xrightarrow{f_1} K_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} K_n$

Persistence module $H_*(K_1) \xrightarrow{f_1} H_*(K_2) \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} H_*(K_n)$

Dual module: $H^*(K_1) \xleftarrow{f_1} H^*(K_2) \xleftarrow{f_2} \dots \xleftarrow{f_{n-1}} H^*(K_n)$

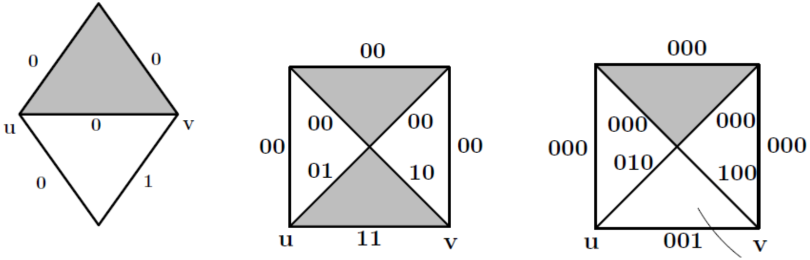
Defn: An **annotation** for $K(p)$, the set of p-simplices of a simplicial complex K is a map $a : K(p) \to \mathbb{Z}_2^g$
Extend linearly to the set of p-chains.

Defn: An annotation is **valid** if

1.) $a : K(p) \to H_p(K) = \mathbb{Z}_2^g$. I.e., $g = rank(H_p(K))$

2.) a induces an isomorphism $a' : H_p(K) \to H_p(K)$.

I.e. $z_1 \in [z_2] \in H_p(K)$ iff $a(z_1) = a(z_2)$.

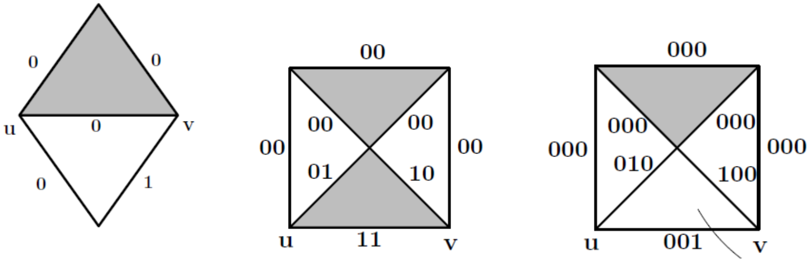


Consider the cochain defined by $\phi_i : K(p) \to \mathbb{Z}_2, \quad \phi_i(\sigma) = a_i(\sigma)$
$i = 1, ..., g$

Prop: TFAE

1.) An annotation $a : K(p) \to \mathbb{Z}_2^g$ is valid.

2.) $\{[\phi_i]\}_{i=1,...,g}$ form a basis for $H^p(K)$



Define cochains $\phi_i : K(p) \to \mathbb{Z}_2, \quad \phi_i(\sigma) = a_i(\sigma), \quad i = 1, ..., g$

Prop: TFAE

1.) An annotation $a : K(p) \to \mathbb{Z}_2^g$ is valid.

2.) $\{[\phi_i]\}_{i=1,...,g}$ form a basis for $H^p(K)$

Pf: (1) $\Rightarrow$ (2): Suppose τ is a $p+1$-simplex. Then $[\partial(\tau)] = [0]$ in H_p.

$\delta_p(\phi_i(\tau)) = \phi_i(\partial(\tau)) = \phi_i(0) = 0$. Thus ϕ_i is a cocycle.

Let V be the vector space generated by $\{[\phi_i]\}_{i=1,...,g}$

Let $\{[z_i]\}_{i=1,...,g}$ be a basis for $H_p(K)$.

Define bilinear form: $\alpha : V \times H_p(K) :\to \mathbb{Z}_2$ by $\alpha([\phi_i], [z_j]) = \phi_i(z_j)$

$[\phi_1(z_j), ..., \phi_g(z_j)] = a(z_j)$. Thus
$[\phi_1(z_j), ..., \phi_g(z_j)] = [\phi_1(z_k), ..., \phi_g(z_k)]$ iff $[z_j] = [z_k]$

Hence $g = rank(H^p) = rank(H_p) = rank < [\phi_i] >$

Define cochains $\phi_i : K(p) \to \mathbb{Z}_2, \quad \phi_i(\sigma) = a_i(\sigma), \quad i = 1, ..., g$

Prop: TFAE

1.) An annotation $a : K(p) \to \mathbb{Z}_2^g$ is valid.

2.) $\{[\phi_i]\}_{i=1,....,g}$ form a basis for $H^p(K)$

Pf: (2) $\Rightarrow$ (1): Let $\{[\phi_i]\}_{i=1,...,g}$ be a basis for $H^p(K)$.

Let $\{[z_i]\}_{i=1,...,g}$ be a basis for $H_p(K)$.

By the universal coefficient theorem:

$H^p(K) \cong Hom(H_p(K), \mathbb{Z}_2)$ by isomorphism sending

cocycle $[\phi_i]$ where $\phi_i : K(p) \to \mathbb{Z}_2$ to

homomorphism $[z_i] \to \phi_i(z_i)$.

Thus $[\phi_1(z_j), ..., \phi_g(z_j)] = [\phi_1(z_k), ..., \phi_g(z_k)]$ iff $[z_j] = [z_k]$

Let $a = [\phi_1, ..., \phi_g]$

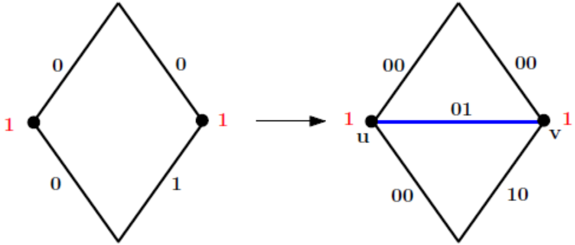
Add simplex, σ, via elementary inclusion.

Case 1: $a(\partial(\sigma)) = 0$. Then σ is a positive simplex.

At time i: $a^i : K^i(p) \to \mathbb{Z}^g$

At time $i+1$: $a^{i+1} : K^{i+1}(p) \to \mathbb{Z}^{g+1}$

$$a^{i+1}(\sigma) = \begin{cases} (a(\sigma), 0) & if \sigma \in K^i(p) \\ (\mathbf{0}, 1) & otherwise \end{cases}$$



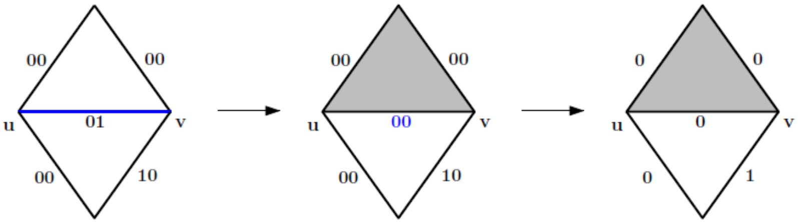
Add simplex, σ, via elementary inclusion.

Case 2: $a(\partial(\sigma)) \neq 0$. Then σ is a negative simplex.

At time i: $a^i : K^i(p) \to \mathbb{Z}^g$

At time $i+1$: $a^{i+1} : K^{i+1}(p) \to \mathbb{Z}^{g-1}$

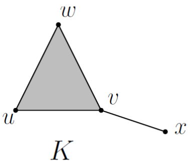
Force $a^{i+1}(\partial\sigma) = \mathbf{0}$



Definition

Stars and Links of $X \subseteq K$:

$$\begin{aligned} \mathtt{St}X &:= \{\sigma \mid \sigma \text{ is a coface of a simplex in } X\} \\ \overline{\mathtt{St}}X &:= \{\text{all faces of simplices in } \mathtt{St}X\} \\ \mathtt{Lk}X &:= \overline{\mathtt{St}}X - \mathtt{St}\overline{X} \end{aligned}$$

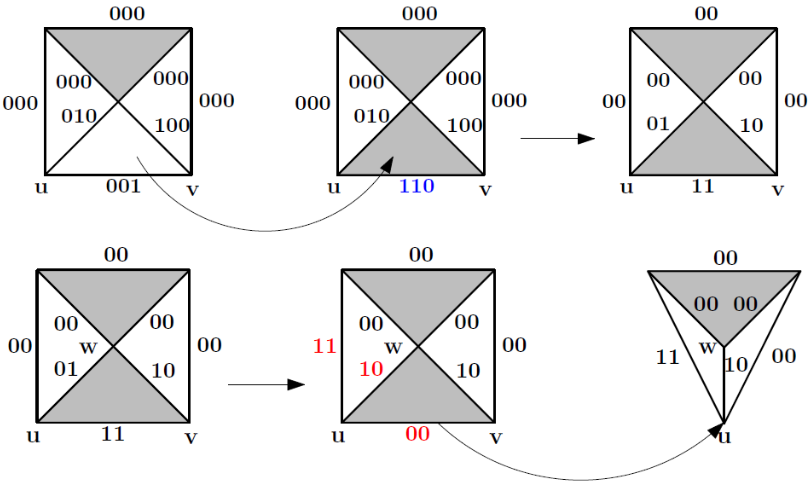


$\overline{\mathtt{St}}v = \{v, vw, vx, uv, uvw, u, w, x$

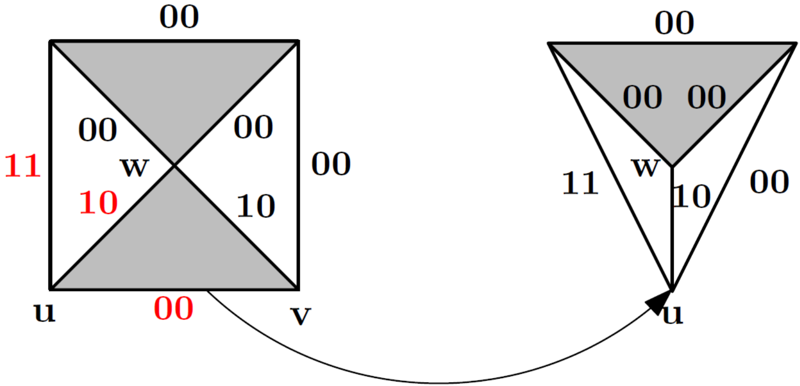
$\mathtt{Lk}v = \{u, w, x, uw\}, \mathtt{Lk}uv = \{w\}$

http://web.cse.ohio-state.edu/ dey.8/talk/simplicial-map/PersistenceForSimplicialMap.pdf

A vertex pair (u, v) statisfies te link condition if the edge $uv \in K_i$ and $Lk(u) \cap Lk(v) = Lk(uv)$.



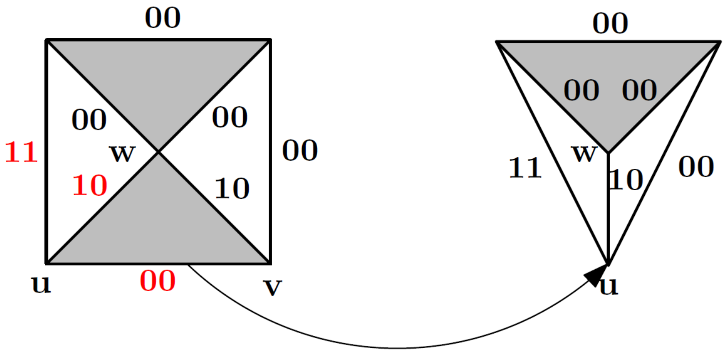
If an elementary collapse, $f_i : K_i \to K_{i+1}$ satisfies the link condition, then K_i and K_{i+1} are homotopy equivalent.



For an elementary collapse $f_i : K_i \to K_{i+1}$:

A simplex $\sigma \in K_i$ is called *vanishing* if $|f_i(\sigma)| = |\sigma| - 1$.

Two simplices are mirror pairs if one contains u and the other contains v and share the rest of the vertices.



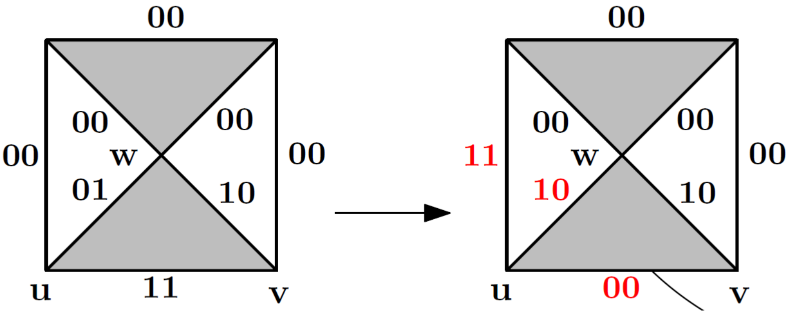
Vanishing simplices: $\{\{u, v\}, \{u, v, w\}\}$

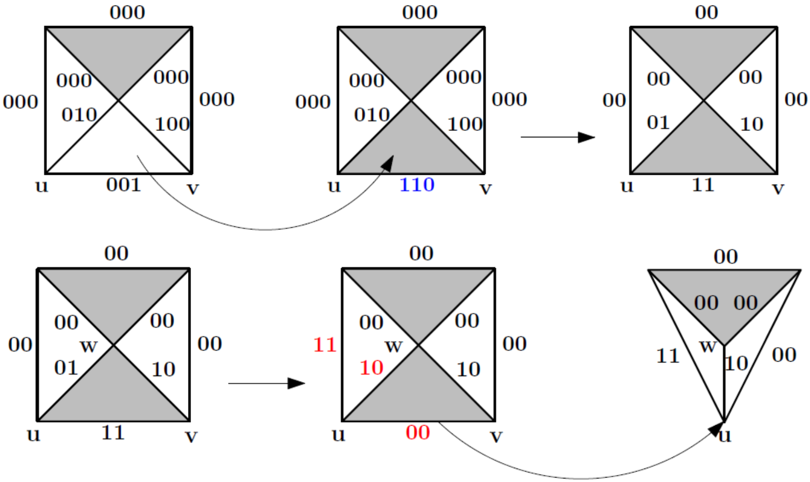
Mirror pairs: $\{\{u\}, \{v\}\}$, $\{\{u, w\}, \{v, w\}\}$

Let $\{[\phi_i]\}_{i=1,\ldots,g}$ be a basis for $H^p(K)$.

Let $\sigma \in K(p)$ and let τ be a $(p-1)$-dimensional face of σ.
Let T = set of cofaces of τ of codimension 1.

Define $\phi_i'(\sigma') = \begin{cases} \phi_i(\sigma') & \text{if } \sigma' \in K_i(p) - T \\ \phi_i(\sigma') + \phi_i(\sigma) & \text{if } \sigma' \in T \end{cases}$

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