Diagonalization Revisted

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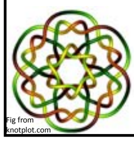
Fig from knotplot.com

Note that multiplying diagonal matrices is easy:

Let $D = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$. Then

$$D^2 = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix} =$$

$$D^k =$$

A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.

Diagonalization has many important applications

It allows one to convert a more complicated problem into a simpler problem.

Example: Calculating A^k when A is diagonalizable.

Application: Calculating A^k.

$$P^{-1}AP \quad = \quad D$$

$k = 1: \; A =$

$k = 2: \; A^2 = PDP^{-1}PDP^{-1}$

$k = 3: \; A^3 = PDP^{-1}PDP^{-1}PDP^{-1}$

Similarly $A^k =$

$$A = PDP^{-1}$$

$$\begin{bmatrix} -1 & 0 \\ -55 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ -55 & 10 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}^3\begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A^3 = PD^3P^{-1}$$

$$\begin{bmatrix} -1 & 0 \\ -55 & 10 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}^3\begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ -55 & 10 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$$

More diagonalization background:

Suppose $AP = PD$ where $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$

I.e., we are assuming A is diagonalizable since

$AP = \quad PD$ implies $P^{-1}AP = D$

Suppose $AP = PD$ where $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$

$AP = A\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \qquad = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} = PD$

Suppose $AP = PD$ where $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$

$AP = A\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 15 & 24 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} = PD$

Hence $A\begin{bmatrix} 1 \\ 3 \end{bmatrix} =$ and $A\begin{bmatrix} 2 \\ 4 \end{bmatrix} =$

Suppose $AP = PD$ where $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$

$AP = A\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 15 & 24 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} = PD$

Hence $A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $A\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Thus an eigenvalue of A = ___ with eigenvector

Another eigenvalue of A = ___ with eigenvector

Thus if $AP = PD$, then
if the diagonal entries of D are $d_1, ..., d_n$

and the i^{th} column of P is an ________________

corresponding to the eigenvalue ______.

Note P is an invertible SQUARE matrix where columns P are ______________ of the matrix A

Suppose $AP = PD$ where $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$

$P^{-1}AP = D$

$AP = A\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 15 & 24 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} = PD$

Hence $A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $A\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Suppose $AP = PD$ where $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$

$P^{-1}AP = D$ $A = PDP^{-1}$

Suppose $AP = PD$ where $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$

$P^{-1}AP = D$ $A = PDP^{-1}$

$$A = PDP^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Suppose $AP = PD$ where $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$

$P^{-1}AP = D$ $A = PDP^{-1}$

$$A = PDP^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 12 \\ 15 & 24 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -10+18 & 5-6 \\ -30+36 & 15-12 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -1 \\ 6 & 3 \end{bmatrix}$$

Suppose $AP = PD$ where $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$

$P^{-1}AP = D$ $A = PDP^{-1}$

$$A = PDP^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ 6 & 3 \end{bmatrix}$$

Check answer:

$$\begin{bmatrix} 8 & -1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8-3 \\ 6+9 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 16-4 \\ 12+12 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

To diagonalize a matrix A:

Step 1: Find eigenvalues: Solve the equation: det $(A - \lambda I) = 0$ for λ.

Step 2: For each eigenvalue, find its corresponding eigenvectors by solving the homogeneous system of equations: $(A - \lambda I)\mathbf{x} = 0$ for **x**.

Case 3a.) IF the geometric multiplicity is LESS then the algebraic multiplicity for at least ONE eigenvalue of A, then A is NOT diagonalizable. (Cannot find square matrix P).

Matrix defective = NOT diagonalizable.

Case 3b.) A is diagonalizable if and only if

geometric multiplicity = algebraic multiplicity

for ALL the eigenvalues of A.

Use the eigenvalues of A to construct the diagonal matrix D

Use the basis of the corresponding eigenspaces for the corresponding columns of P. (NOTE: P is a SQUARE matrix).

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$$A = \begin{bmatrix} 4 & 1 \\ 7 & -2 \end{bmatrix}$$

For more complicated example, see video 4: Eigenvalue/Eigenvector Example & video 5: Diagonalization

Step 1: Find eigenvalues: Solve the equation: det (A – λI) = 0 for λ.

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 7 & -2-\lambda \end{vmatrix} = (4-\lambda)(-2-\lambda) - 7$$

$$= -8 - 2\lambda + \lambda^2 - 7 = \lambda^2 - 2\lambda - 15$$

$$= (\lambda + 3)(\lambda - 5) = 0$$

characteristic equation: $(\lambda + 3)(\lambda - 5) = 0$

λ = -3 : algebraic multiplicity =
geometric multiplicity =
dimension of eigenspace =

λ = 5 : algebraic multiplicity
geometric multiplicity
dimension of eigenspace

1 ≤ geometric multiplicity ≤ algebraic multiplicity

characteristic equation: $(\lambda + 3)(\lambda - 5) = 0$

λ = -3 : algebraic multiplicity = 1
geometric multiplicity = 1
dimension of eigenspace = 1

Matrix is not defective.

λ = 5 : algebraic multiplicity = 1
geometric multiplicity = 1
dimension of eigenspace = 1

1 ≤ geometric multiplicity ≤ algebraic multiplicity

characteristic equation: $(\lambda + 3)(\lambda - 5) = 0$

λ = -3 : algebraic multiplicity = 1
geometric multiplicity = 1
dimension of eigenspace = 1

Matrix is not defective.

Thus A is diagonalizable

λ = 5 : algebraic multiplicity = 1
geometric multiplicity = 1
dimension of eigenspace = 1

1 ≤ geometric multiplicity ≤ algebraic multiplicity

characteristic equation: $(\lambda + 3)(\lambda - 5) = 0$

λ = -3 : algebraic multiplicity = 1
geometric multiplicity = 1
dimension of eigenspace = 1

Matrix is not defective.

Thus A is diagonalizable

λ = 5 : algebraic multiplicity = 1
geometric multiplicity = 1
dimension of eigenspace = 1

$$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

1 ≤ geometric multiplicity ≤ algebraic multiplicity

Find eigenvectors to create P

Nul(A + 3I) = eigenspace corresponding to eigenvalue λ = -3 of A

$$A - (-3)I = \begin{bmatrix} 4+3 & 1 \\ 7 & -2+3 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 7 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 7 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/7 \\ 0 & 0 \end{bmatrix}$$

$$A-(-3)I \sim \begin{bmatrix} 1 & 1/7 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7}x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ 1 \end{bmatrix} x_2$$

$$A-(-3)I \sim \begin{bmatrix} 1 & 1/7 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7}x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ 1 \end{bmatrix} x_2$$

Basis for eigenspace corresponding to λ = -3:

$$\left\{ \begin{bmatrix} -\frac{1}{7} \\ 1 \end{bmatrix} \right\}$$

$$A-(-3)I \sim \begin{bmatrix} 1 & 1/7 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7}x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ 1 \end{bmatrix} x_2$$

Basis for eigenspace corresponding to λ = -3:

$$\left\{ \begin{bmatrix} -\frac{1}{7} \\ 1 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} -1 \\ 7 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ -7 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} -2 \\ 14 \end{bmatrix} \right\}$$

Find eigenvectors to create P

Nul(A - 5I) = eigenspace corresponding to eigenvalue λ = 5 of A

$$A-5I = \begin{bmatrix} 4-5 & 1 \\ 7 & -2-5 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$

Basis for eigenspace corresponding to λ = 5:

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} \pi \\ \pi \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

$$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

Basis for eigenspace corresponding to λ = -3: $\left\{ \begin{bmatrix} -1 \\ 7 \end{bmatrix} \right\}$

Basis for eigenspace corresponding to λ = 5: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$P =$

$$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

Basis for eigenspace corresponding to λ = -3: $\left\{ \begin{bmatrix} -1 \\ 7 \end{bmatrix} \right\}$

Basis for eigenspace corresponding to λ = 5: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$P = \begin{bmatrix} -1 & 1 \\ 7 & 1 \end{bmatrix}$$

$P^{-1}AP = D$

Note we want $P = \begin{bmatrix} -1 & 1 \\ 7 & 1 \end{bmatrix}$ to be invertible.

Note P is invertible if and only if the columns of P are linearly independent.

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Thm: Suppose λ_i, $i = 1, \ldots, n$ are DISTINCT eigenvalues of a matrix A. If $\mathcal{B}_i$ is a basis for the eigenspace corresponding to λ_i, then $\mathcal{B} = \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n$ is linearly independent.

Note: You can easily check your answer.

$P^{-1}AP = D$ implies $AP = PD$

$$AP = \begin{bmatrix} 4 & 1 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -21 & 5 \end{bmatrix}$$

$$PD = \begin{bmatrix} -1 & 1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -21 & 5 \end{bmatrix}$$

Diagonalize $A = \begin{bmatrix} 4 & 1 \\ 7 & -2 \end{bmatrix}$

Note there are many correct answers.

$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$ & $P = \begin{bmatrix} -1 & 1 \\ 7 & 1 \end{bmatrix}$	$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$ & $P = \begin{bmatrix} 10 & \pi \\ -70 & \pi \end{bmatrix}$
$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$ & $P = \begin{bmatrix} -2 & 1 \\ 14 & 1 \end{bmatrix}$	$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$ & $P = \begin{bmatrix} -1 & 1 \\ 7 & 1 \end{bmatrix}$
$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$ & $P = \begin{bmatrix} -1 & 4 \\ 7 & 4 \end{bmatrix}$	$D = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$ & $P = \begin{bmatrix} 1 & -1 \\ 1 & 7 \end{bmatrix}$

Diagonalize $A = \begin{bmatrix} 4 & 1 \\ 7 & -2 \end{bmatrix}$

Note there are many correct answers.

$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$ & $P = \begin{bmatrix} -1 & 1 \\ 7 & 1 \end{bmatrix}$	$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$ & $P = \begin{bmatrix} 10 & \pi \\ -70 & \pi \end{bmatrix}$
$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$ & $P = \begin{bmatrix} -2 & 1 \\ 14 & 1 \end{bmatrix}$	$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$ & $P = \begin{bmatrix} -1 & 1 \\ 7 & 1 \end{bmatrix}$
$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$ & $P = \begin{bmatrix} -1 & 4 \\ 7 & 4 \end{bmatrix}$	$D = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$ & $P = \begin{bmatrix} 1 & -1 \\ 1 & 7 \end{bmatrix}$

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