Section 1.4

$$\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix}$$ Aug

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$ lin comb

$$\boxed{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$ matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Matrices as linear combinations:

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \cdot \\ \cdot \\ \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \dots, a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} x_2 + \dots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

1.7: Linear Independence.

Defn: The set of vectors $\{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}\}$ is **linearly independent** if and only if the equation $c_1\mathbf{a_1} + c_2\mathbf{a_2} + ... + c_n\mathbf{a_n} = \mathbf{0}$ has only the trivial solution.

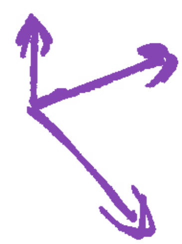
homog

Defn: If $\mathcal{S}$ is not linearly independent, then it is **linearly dependent**.

Is $\left\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix}\right\}$ is linearly independent? $\boxed{\text{NO}}$

$$\begin{bmatrix} 9 \\ 7 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ 8 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ -5 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 9x_1 + 4x_2 + 3x_3 &= 0 \\ 7x_1 + 8x_2 - 5x_3 &= 0 \end{aligned}$$



$\infty$ # of sol'n

$$\left[\underbrace{\begin{matrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{matrix}}_{\text{coef}} \;\middle|\; \begin{matrix} 0 \\ 0 \end{matrix}\right]$$

$$\underbrace{\begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3 columns > 2 rows
$\Rightarrow$ at least 1 free variable
$\Rightarrow \infty$ # of sol'ns

$\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ is linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has exactly one solution (pivot in every column of echelon form of coefficient matrix $A$).

$\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ is linearly dependent if and only if $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions. (at least one free variable)

Thm: Let $S = \{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}\}$ be a set of vectors in $R^k$. Then $S$ is linearly dependent if and only if the vector equation $c_1\mathbf{a_1} + c_2\mathbf{a_2} + ... + c_n\mathbf{a_n} = \mathbf{0}$ has an infinite number of solutions.

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right\}$ linearly independent?

$$\begin{bmatrix} 1 & 4 & 5 & 0 \\ 2 & 5 & 7 & 0 \\ 3 & 6 & 9 & 0 \end{bmatrix}$$

drop constants' column since homogeneous equation & row ops don't affect 0's

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - 1\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ lin dep

$$0\begin{bmatrix} \ \end{bmatrix} + 0\begin{bmatrix} \ \end{bmatrix} + 0\begin{bmatrix} \ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\infty$ # of sol'ns $\Rightarrow$ lin dep

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$$\begin{bmatrix} 1 & 4 & 5 \\ 2^{-2} & 5^{-8} & 7^{-10} \\ 3^{-3} & 6^{-12} & 9^{-15} \end{bmatrix} \xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & -6^{+6} & -6^{+6} \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 4 & \boxed{5} & 0 \\ 0 & -3 & \boxed{-3} & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

free variable

$\infty$ # of sol'ns to homog

$\Rightarrow$ lin dep

$\left\{ \begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\}$ is linearly dependent since

$$\begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has an infinite number of solutions

$$\left( \text{In particular } \begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{3}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

or equivalently,

$$1 \begin{bmatrix} 9 \\ 7 \end{bmatrix} - \left(\frac{3}{2}\right) \begin{bmatrix} 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or equivalently,

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} - (3/2) \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

can find non-trivial sol'n to homog eq

or alternatively,

3 vectors in $R^2$ cannot be linearly independent.

. more columns than rows $\Rightarrow$ free variables

$[\begin{smallmatrix}9\\7\end{smallmatrix}], [\begin{smallmatrix}4\\8\end{smallmatrix}], [\begin{smallmatrix}3\\-5\end{smallmatrix}]$

FYI: Is $\{9 + 7t, 4 + 8t, 3 - 5t\}$ linearly independent? NO

1st coordinate 2nd coordinate

Thm: Let $S$ be a set of $n$ vectors in $R^k$ where $n > k$. Then $S$ is linearly dependent.

more columns than rows $\Rightarrow$ free variables $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix}$

Thm: A set of vectors is linearly dependent if one of the vectors can be written as a linear combination of the other vectors.

A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the other vectors.

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Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \right\}$ linearly independent? NO

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ linearly independent? YES

only 2 vectors $\Rightarrow$ can easily determine if one is a lin comb of other

Is $\left\{ \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ linearly independent? YES

$$\text{Is } \left\{ \begin{bmatrix}1\\2\\3\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\\2\end{bmatrix}, \begin{bmatrix}5\\7\\9\\1\end{bmatrix}, \begin{bmatrix}6\\2\\1\\5\end{bmatrix}, \begin{bmatrix}4\\3\\4\\6\end{bmatrix}, \begin{bmatrix}1\\4\\3\\7\end{bmatrix} \right\} \text{ linearly independent? } \text{NO}$$

6 vectors in $\mathbb{R}^4$

$$\text{Is } \left\{ \boxed{\begin{bmatrix}0\\0\\0\end{bmatrix}}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}4\\-3\\0\end{bmatrix} \right\} \text{ linearly independent? } \text{NO}$$

$$5\vec{0} + 0\vec{v} + 0\vec{w} = \vec{0}$$

$\Rightarrow$ lin dep

$$\text{Is } \left\{ \begin{bmatrix}0\\-1\\-2\end{bmatrix} \right\} \text{ linearly independent? } \text{YES}$$

$$0\begin{bmatrix}0\\-1\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\text{Is } \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}5\\7\\9\end{bmatrix} \right\} \text{ linearly independent? } \text{NO}$$

If not obvious $\Rightarrow$ do row ops

$$\begin{bmatrix}1&4&5\\2&5&7\\3&6&9\end{bmatrix} \to \begin{bmatrix}1&4&5\\0&-3&-3\\0&-6&-6\end{bmatrix} \to \begin{bmatrix}1&4&\boxed{5}\\0&-3&-3\\0&0&0\end{bmatrix}$$

## 2.1: Operations on Matrices

$A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$.

Defn: Two matrices $A$ and $B$ are equal, if they have the same dimension and $a_{ij} = b_{ij}$ for all $i = 1, ..., n$, $j = 1, ..., m$

Defn: $A + B = (a_{ij} + b_{ij})$.

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$

Defn: $cA = (ca_{ij})$.

$3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) \\ 3(3) & 3(4) \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

Defn: $-B = (-1)B$.

Defn: $A - B = A + (-B)$.

Defn: The zero matrix $= 0 = (a_{ij})$ where $a_{ij} = 0$ for all $i, j$.

Ex: $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, ...

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{0}$ vector

0 matrix

Defn: The identity matrix $= I = (a_{ij})$ where $a_{ij} = 0$ for all $i \neq j$ and $a_{ii} = 1$ for all $i$ and $I$ is a square matrix

Ex: $[\,1\,]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, ...

$A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$.

rows columns rows columns

Suppose $A$ is an $m \times k$ matrix, $B$ is an $k \times n$.
$AB = C$ where

$$c_{ij} = row(i) \ of \ A \cdot column(j) \ of \ B$$
$$= a_{i1}b_{1j} + a_{i2}b_{2k} + ... + a_{ir}b_{rk}.$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} = \begin{bmatrix} 1(5)+2(8) & 6+18 & 7+20 \\ 3(5)+4(8) & 18+36 & 21+40 \\ 0(5)+1(8) & 0+9 & 0+10 \\ 1(5)+0(8) & 6+0 & 7+0 \end{bmatrix} = \begin{bmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \\ 8 & 9 & 10 \\ 5 & 6 & 7 \end{bmatrix}$$

$4 \times 2$ $\quad 2 \times 3$ $\quad 4 \times 3$

$$\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \text{NOT DEFINED}$$

$2 \times 3$ $\quad 4 \times 2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$

$2 \times 2$ $\quad 2 \times 3$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$

$$(2 \times 3) \quad (3 \times 3)$$

$$I_{3 \times 3}$$

$$I_{3 \times 3} \neq I_{2 \times 2}$$

$$\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{NOT DEFINED}$$

$$2 \times 3 \qquad 2 \times 2$$

L R

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Matrix multiplication is <u>NOT</u> commutative

$$AB \neq BA$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \neq \begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}$$

undefined

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 4 \end{bmatrix}$$

Matrix mult is linear

$$A(s\vec{u} + t\vec{v})$$

vector or matrix

$$sAU + tAV$$

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$$A\vec{u} = \vec{0} \quad \& \quad A\vec{v} = 0$$

$$A(s\vec{u} + t\vec{v}) = sA\vec{u} + tA\vec{v}$$

$$= \vec{0} + \vec{0} = \vec{0}$$

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$$A\vec{p} = \vec{b}$$