Math 2550 Matrix Algebra Exam 3 **Form C**
Dec. 4, 2014

[18] 1.) Find the characteristic equation and diagonalize $A = \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix}$

NOTE: A is clearly not invertible (since det(A) = 0 or equivalently columns are linearly dependent or equivalently (since A square) rows or linearly dependent). Thus 0 is an eigenvalue of A.

Find eigenvalues:
$$det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 8 \\ 8 & 16-\lambda \end{vmatrix} = (4-\lambda)(16-\lambda) - 64 = 64 - 20\lambda + \lambda^2 - 64 = \lambda^2 - 20\lambda = \lambda(\lambda-20) = 0$$

Characteristic equation of $A = \underline{\lambda(\lambda-20) = 0}$.

Find eigenvectors:

$$\lambda = 0: \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \text{ implies } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

$$\text{Check: } \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda = -20: \begin{bmatrix} -16 & 8 \\ 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \text{ implies } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} x_2.$$

Thus $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ is an e. vector of A. Hence $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is also an e-vector of A.

$$\text{Check: } \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \end{bmatrix} = 20\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$detP = -4 - 1 = -5$

$$P = \underline{\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}} \qquad D = \underline{\begin{bmatrix} 0 & 0 \\ 0 & 20 \end{bmatrix}} \qquad P^{-1} = \underline{\begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}}.$$

Note: if you forgot the formula for P^{-1}, you could (either derive it or) notice that A is symmetric and P is orthogonal. Thus we can normalize the columns of P so that for the new orthonormal P, $P^{-1} = P^T$ (since columns of new P are orthonormal). Thus alternative answer:

$$P = \underline{\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}} \qquad D = \underline{\begin{bmatrix} 0 & 0 \\ 0 & 20 \end{bmatrix}} \qquad P^{-1} = \underline{\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}}.$$

[16] 2.) Use Gram-Schmidt to find the QR factorization of $M = \begin{bmatrix} 1 & 6 \\ 2 & 6 \\ 2 & 0 \end{bmatrix}$.

Note one can work with scaled vectors to find Q (think of the pictures relating to orthogonal projection and orthogonal component), but not R. For those not comfortable with scaling, we will work with the vectors as given.

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 1+4+4 = 9 \qquad \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix} = 6+12+0 = 18$$

$$proj_{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix} = \frac{18}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

Orthogonal component = $\begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$

Normalize: length of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \sqrt{9} = 3$ length of $\begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix} = \sqrt{16+4+16} = 6$

Thus $Q = \begin{bmatrix} \frac{1}{3} & \frac{4}{6} \\ \frac{2}{3} & \frac{2}{6} \\ \frac{2}{3} & -\frac{4}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$.

$M = QR$ implies $Q^T M = Q^T QR = R$

Thus $R = Q^T M = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 2 & 6 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+4+4}{3} & 2+4+0 \\ \frac{2-2-0}{3} & 4+2+0 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 0 & 6 \end{bmatrix}$

NOTE: Columns of Q are orthonormal and R is upper triangular.

$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$ $R = \begin{bmatrix} 3 & 6 \\ 0 & 6 \end{bmatrix}$

Part 2: Multiple Choice (multiple choice problems are worth 6 points each).

Problem 1. There are an infinite number of eigenvectors that correspond to a particular eigenvalue of A

- A. True
- B. False

Problem 2. The vector $\mathbf{0}$ is an eigenvector of A if and only if $\det(A) = 0$.

- A. True
- B. False

Problem 3. If the characteristic polynomial of $A = (\lambda-7)^1(\lambda+7)^2(\lambda+8)^6$, then the algebraic multiplicity of $\lambda = -7$ is

- A. 0
- B. 1
- C. 2
- D. 3
- E. 0 or 1
- F. 0 or 2
- G. 1 or 2
- H. 0, 1, or 2
- I. 0, 1, 2, or 3
- J. none of the above

Problem 4. Let $A = \begin{bmatrix} 4 & 5 & 10 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$. Is A = diagonalizable?

- A. yes
- B. no
- C. none of the above

Problem 5. If A is diagonalizable, then A is symmetric.

- A. True
- B. False

Problem 6. Suppose $A = PDP^{-1}$ where D is a diagonal matrix. Suppose also the d_{ii} are the diagonal entries of D. If $P = [\ \vec{p}_1\ \vec{p}_2\ \vec{p}_3\ ]$ and $d_{11} = d_{22}$, then $2\vec{p}_1 + 5\vec{p}_2$ is an eigenvector of A

- A. True
- B. False

Problem 7. If the characteristic polynomial of $A = (\lambda-7)^9(\lambda-3)^2(\lambda+2)^4$, then the geometric multiplicity of $\lambda = 3$ is

- A. 0
- B. 1
- C. 2
- D. 3
- E. 0 or 1
- F. 0 or 2
- G. 1 or 2
- H. 0, 1, or 2
- I. 0, 1, 2, or 3
- J. none of the above

Problem 8. If x is in a subspace W, then $x - \text{proj}_W(x)$ is not zero.

- A. True
- B. False

Problem 9. Let $A = \begin{bmatrix} -4 & 5 & 10 \\ 0 & -3 & 2 \\ 0 & 0 & -4 \end{bmatrix}$. Is A = diagonalizable?

- A. yes
- B. no
- C. none of the above

Problem 10. Suppose $A \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \\ -1 \end{bmatrix}$. Then an eigenvalue of A is

- A. -4
- B. -3
- C. -2
- D. -1
- E. 0
- F. 1
- G. 2
- H. 3
- I. 4
- J. none of the above

Problem 11. Suppose the orthogonal projection of $\begin{bmatrix} 13 \\ 3 \\ -4 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$ is (z_1, z_2, z_3). Then $z_1 =$

- A. -4
- B. -3
- C. -2
- D. -1
- E. 0
- F. 1
- G. 2
- H. 3
- I. 4
- J. none of the above