Math 2550 Matrix Algebra Exam 3 **Form D**
Dec. 4, 2014

[18] 1.) Find the characteristic equation and diagonalize $A = \begin{bmatrix} -2 & 6 \\ 6 & -18 \end{bmatrix}$

NOTE: A is clearly not invertible (since det(A) = 0 or equivalently columns are linearly dependent or equivalently (since A square) rows or linearly dependent). Thus 0 is an eigenvalue of A.

$Ax = 0$
$A\vec{x} = 0\vec{x}$

Find eigenvalues:

$$det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 6 \\ 6 & -18-\lambda \end{vmatrix} = (-2-\lambda)(-18-\lambda) - 36 = 36 + 20\lambda + \lambda^2 - 36 = \lambda^2 + 20\lambda = \lambda(\lambda+20) = 0$$

Characteristic equation of $A = \underline{\lambda(\lambda+20) = 0}$.

Find eigenvectors:

$$\lambda = 0: \begin{bmatrix} -2 & 6 \\ 6 & -18 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \text{ implies } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} x_2$$

$$\text{Check: } \begin{bmatrix} -2 & 6 \\ 6 & -18 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\lambda = -20: \begin{bmatrix} 18 & 6 \\ 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix} \text{ implies } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} x_2.$$

Thus $\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$ is an e. vector of A. Hence $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is also an e-vector of A.

$$\text{Check: } \begin{bmatrix} -2 & 6 \\ 6 & -18 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 20 \\ -60 \end{bmatrix} = -20 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$detP = 9 + 1 = 10$

$$P = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & -20 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix}.$$

$PP^{-1} = I$ ✓

Note: if you forgot the formula for P^{-1}, you could (either derive it or) notice that A is symmetric and P is orthogonal. Thus we can normalize the columns of P so that for the new orthonormal P, $P^{-1} = P^T$ (since columns of new P are orthonormal). Thus alternative answer:

$$P = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & -20 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

A sym $\Rightarrow$ P orthog ~~col~~ e.vectors from different e.space are $\perp$

[16] 2.) Use Gram-Schmidt to find the QR factorization of $M = \begin{bmatrix} 6 & 3 \\ 6 & 0 \\ 3 & 3 \end{bmatrix}$.

Note one can work with scaled vectors to find Q (think of the pictures relating to orthogonal projection and orthogonal component), but not R. For those not comfortable with scaling, we will work with the vectors as given.

$$\begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} = 36+36+9 = 81 \qquad \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = 18+0+9 = 27$$

$$proj_{\begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = \frac{27}{81}\begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{Orthogonal component} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$\text{Normalize: length of } \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} = \sqrt{81} = 9 \qquad \text{length of } \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \sqrt{1+4+4} = 3$$

$$\text{Thus } Q = \begin{bmatrix} \frac{6}{9} & \frac{1}{3} \\ \frac{6}{9} & -\frac{2}{3} \\ \frac{3}{9} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

$M = QR$ implies $Q^T M = Q^T QR = R$

$$\text{Thus } R = Q^T M = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 6 & 0 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 4+4+1 & 2+0+1 \\ 2-4+2 & 1-0+2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 0 & 3 \end{bmatrix}$$

NOTE: Columns of Q are orthonormal and R is upper triangular.

$$Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \qquad R = \begin{bmatrix} 9 & 3 \\ 0 & 3 \end{bmatrix}$$

$\perp$ perpendicular

1. (1 pt) local/Library/UI/Fall14/quiz2_9.pg
Supppose A is an invertible $n \times n$ matrix and v is an eigenvector of A with associated eigenvalue -5. Convince yourself that v is an eigenvector of the following matrices, and find the associated eigenvalues:

1. A^8, eigenvalue =

- A. 16
- B. 81
- C. 125
- D. 216
- E. 1024
- F. 390625
- G. 2000
- H. None of those above

$A\vec{v} = \lambda\vec{v}$

$A^2\vec{v} = A(A\vec{v})$

$= A(\lambda\vec{v})$

$= \lambda(A\vec{v})$

$= \lambda(\lambda\vec{v})$

$= \lambda^2\vec{v}$

$A^8\vec{v} = \lambda^8\vec{v}$

2. A^{-1}, eigenvalue =

- A. -0.5
- B. -0.333
- C. -0.2
- D. -0.125
- E. 0
- F. 0.125
- G. 0.333
- H. 0.5
- I. None of those above

3. $A - 4I_n$, eigenvalue =

- A. -8
- B. -4
- C. -5
- D. 0
- E. 2
- F. 4
- G. -9
- H. 10
- I. None of those above

$(A - 4I_n)\vec{v}$

$A\vec{v} - 4I_n\vec{v}$

$= \lambda\vec{v} - 4\vec{v}$

$= (\lambda - 4)\vec{v}$

4. $8A$, eigenvalue =

- A. -36
- B. -28
- C. -40
- D. -12
- E. 0
- F. 24
- G. 36
- H. None of those above

2. (1 pt) local/Library/UI/Fall14/quiz2_10.pg
If $v_1 = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are eigenvectors of a matrix A corresponding to the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$, respectively, then

a. $A(v_1 + v_2) =$

$A\vec{v}_1 + A\vec{v}_2$

- A. $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$
- B. $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$
- C. $\begin{bmatrix} -6 \\ 5 \end{bmatrix}$
- D. $\begin{bmatrix} 10 \\ 6 \end{bmatrix}$
- E. $\begin{bmatrix} 12 \\ 4 \end{bmatrix}$
- F. $\begin{bmatrix} -6 \\ 10 \end{bmatrix}$
- G. None of those above

b. $A(-3v_1) =$

- A. $\begin{bmatrix} -12 \\ -12 \end{bmatrix}$
- B. $\begin{bmatrix} -2 \\ 8 \end{bmatrix}$
- C. $\begin{bmatrix} -6 \\ 4 \end{bmatrix}$
- D. $\begin{bmatrix} 10 \\ 6 \end{bmatrix}$
- E. $\begin{bmatrix} 30 \\ -30 \end{bmatrix}$
- F. $\begin{bmatrix} 12 \\ 4 \end{bmatrix}$
- G. $\begin{bmatrix} -6 \\ 10 \end{bmatrix}$
- H. None of those above

Problem 1. **1.** (1 pt) Library/TCNJ/TCNJ_MatrixEquations-/problem4.pg

Let $A = \begin{bmatrix} -5 & 1 & 3 \\ -3 & 3 & 5 \\ 2 & 4 & -2 \end{bmatrix}$ and $x = \begin{bmatrix} -4 \\ -5 \\ 3 \end{bmatrix}$.

[?] 1. What does Ax mean?

Problem 2. **2.** (1 pt) local/Library/UI/Fall14/HW7_12.pg

Suppose that A is a 5×9 matrix which has a null space of dimension 6. The rank of A=

- A. -4
- B. -3
- C. -2
- D. -1
- E. 0
- F. 1
- G. 2
- H. 3
- I. 4
- J. none of the above

Problem 3. **3.** (1 pt) Library/WHFreeman/Holt_linear_algebra-/Chaps_1-4/2.3.42.pg

Let A be a matrix with more columns than rows.
Select the best statement.

- A. The columns of A are linearly independent, as long as no column is a scalar multiple of another column in A
- B. The columns of A are linearly independent, as long as they does not include the zero vector.
- C. The columns of A could be either linearly dependent or linearly independent depending on the case.
- D. The columns of A must be linearly dependent.
- E. none of the above

Problem 4. Suppose $A\vec{x} = \vec{0}$ has an infinite number of solutions, then given a vector $\vec{b}$ of the appropriate dimension, $A\vec{x} = \vec{b}$ has

- A. No solution
- B. Unique solution
- C. Infinitely many solutions
- D. at most one solution
- E. either no solution or an infinite number of solutions
- F. either a unique solution or an infinite number of solutions
- G. no solution, a unique solution or an infinite number of solutions, depending on the system of equations
- H. none of the above

Problem 5. Suppose A is a square matrix and $A\vec{x} = \vec{0}$ has an infinite number of solutions, then given a vector $\vec{b}$ of the appropriate dimension, $A\vec{x} = \vec{b}$ has

- A. No solution
- B. Unique solution
- C. Infinitely many solutions
- D. at most one solution
- E. either no solution or an infinite number of solutions
- F. either a unique solution or an infinite number of solutions
- G. no solution, a unique solution or an infinite number of solutions, depending on the system of equations
- H. none of the above

Problem 6. The vector $\vec{b}$ is in $ColA$ if and only if $A\vec{v} = \vec{b}$ has a solution

- A. True
- B. False

Problem 7. The vector $\vec{v}$ is in $NulA$ if and only if $A\vec{v} = \vec{0}$

- A. True
- B. False

Problem 8. If $\vec{x}_1$ and $\vec{x}_2$ are solutions to $A\vec{x} = \vec{0}$, then $-2\vec{x}_1 + 2\vec{x}_2$ is also a solution to $A\vec{x} = \vec{0}$.

- A. True
- B. False

Problem 9. If $\vec{x}_1$ and $\vec{x}_2$ are solutions to $A\vec{x} = \vec{b}$, then $-7\vec{x}_1 + 3\vec{x}_2$ is also a solution to $A\vec{x} = \vec{b}$.

$A\vec{x}_1 = \vec{b}$ $\quad$ $A\vec{x}_2 = \vec{b}$

- A. True
- B. False

$A(-7\vec{x}_1 + 3\vec{x}_2) = -7A\vec{x}_1 + 3A\vec{x}_2$

$= -7\vec{b} + 3\vec{b} = 4\vec{b}$

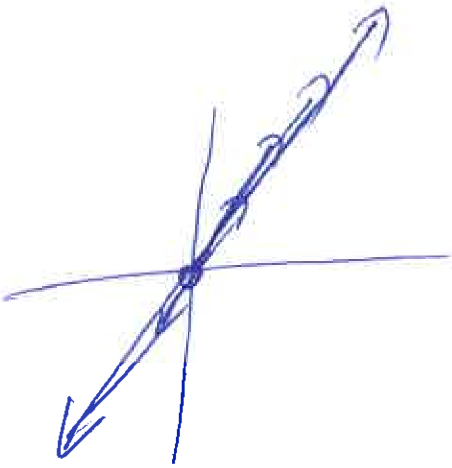
$\neq \vec{b}$

Homog Eqn

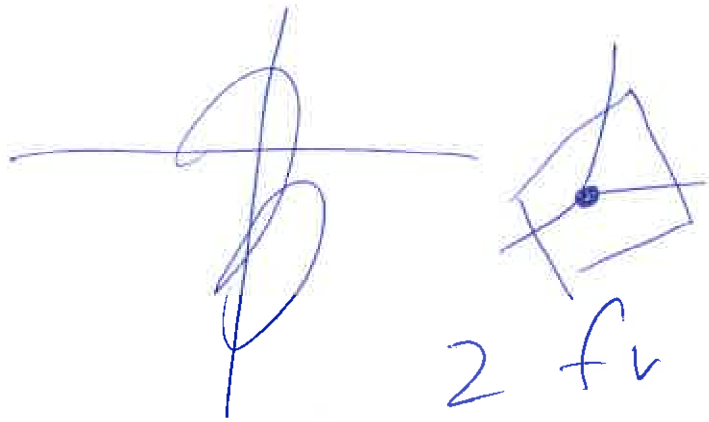
$A\vec{x} = \vec{0}$

suppose V_i are sol'n to $A\vec{x} = \vec{0}$

$\Rightarrow \sum c_i \vec{V_i}$ is also a sol'n to $A\vec{x} = 0$



1 free variable



2 fv

Non homog eqn

$A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$

If $\sum c_i \vec{v_i}$ is the general sol'n to homog eqn $A\vec{x} = 0$

and $\vec{u}$ is a sol'n to nonhomog $A\vec{x} = \vec{b}$

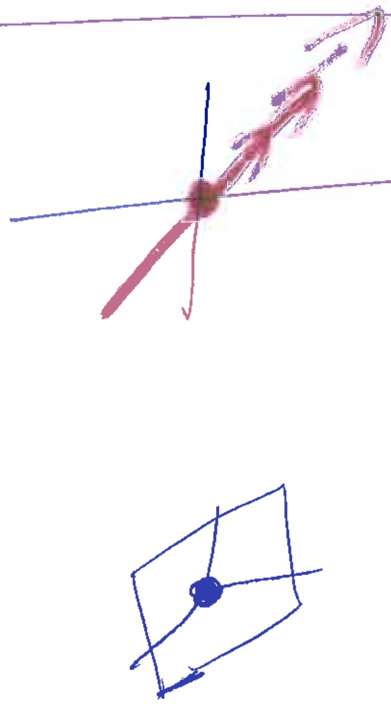
$\Rightarrow$ General sol'n to $A\vec{x} = \vec{b}$

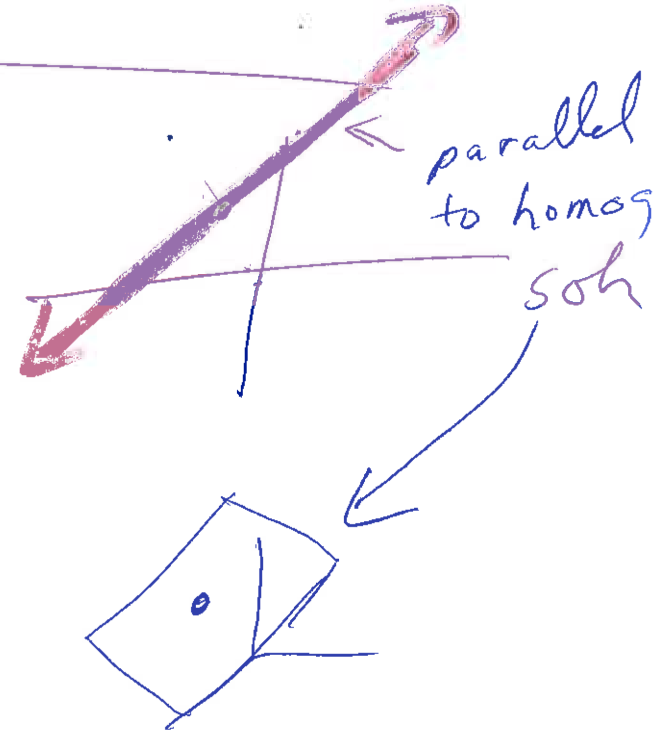
$$\vec{u} + \left(\sum c_i \vec{v_i}\right)$$

homog | non homog

1 free var

2 free var





homog $\xrightarrow{\text{parallel copy}}$ non homog

$$\sum_{i=1}^{K} c_i v_i \longrightarrow \vec{u} + \sum_{i=1}^{K} c_i \vec{v_i}$$

K free variables

$\cancel{A^{-1}A}x = A^{-1}b \Rightarrow \vec{x} = A^{-1}\vec{b}$ ② solve using inverse

③ solve using Cramer's Rule

① or solve using REF

Thm 8′: If A is a SQUARE $n \times n$ matrix, then the following are equivalent.

a.) A is invertible. $\det A \neq 0$

b.) The row-reduced echelon form of A is I_n, the identity matrix.

c.) An echelon form of A has n leading entries [I.e., every column of an echelon form of A is a leading entry column – no free variables]. (A square => A has leading entry in every column if and only if A has leading entry in every row).

d.) The column vectors of A are linearly independent.

e.) $Ax = 0$ has only the trivial solution.

f.) $Ax = b$ has at most one sol'n for any b.

g.) $Ax = b$ has a unique sol'n for any b.

h.) $Ax = b$ is consistent for every $n \times 1$ matrix b.

i.) $Ax = b$ has at least one sol'n for any b.

j.) The column vectors of A span R^n. [every vector in R^n can be written as a linear combination of the columns of A].

No free variables

all column are pivot columns

SQUARE

k.) There is a square matrix C such that $CA = I$.

l.) There is a square matrix D such that $AD = I$.

SQUARE

every row is a pivot row

m.) A^T is invertible.

n.) A is expressible as a product of elementary matrices.

o.) The column vectors of A form a basis for R^n. [every vector in R^n can be written uniquely as a linear combination of the columns of A].

p.) Col $A = R^n$.

q.) dim Col $A = n$.

r.) rank of $A = n$.

no row of all zero's in EF

s.) Nul $A = \{0\}$,

t.) dim Nul $A = 0$.

u.) A has nullity 0. ← # of free variables

v.) $\boxed{\lambda = 0 \text{ is NOT an eigenvalue of } A}$

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

3.3: Cramer's Rule, Adjoint, Inverses, Area

Defn: Let $A_i(\mathbf{b})$ = the matrix derived from A by replacing the i^{th} column of A with $\mathbf{b}$.

Cramer's Rule: Suppose $A\mathbf{x} = \mathbf{b}$ where A is an $n \times n$ matrix such that $\boxed{det A \neq 0.}$ Then

$$x_i = \frac{det A_i(\mathbf{b})}{det A}.$$

unique sol'n

Solve the following using Cramer's rule:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (3)(2) = 4 - 6 = -2$ $= det\ A$

$det \begin{bmatrix} 5 & 2 \\ 6 & 4 \end{bmatrix} = (5)(4) - (6)(2) = 20 - 12 = 8$ $= det\ A_1 \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$det \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix} = (1)(6) - (3)(5) = 6 - 15 = -9$ $= det\ A_2 \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

A_2

Thus $x_1 = \frac{8}{-2} = -4, \quad x_2 = \frac{-9}{-2} = \frac{9}{2}.$

$= x_1 = \dfrac{det\ A_1 \begin{bmatrix} 5 \\ 6 \end{bmatrix}}{det\ A} = \dfrac{8}{-2} = -4$

$x_2 = \dfrac{det\ A_2 \begin{bmatrix} 5 \\ 6 \end{bmatrix}}{det\ A} = \dfrac{-9}{-2} = \dfrac{9}{2}$

Defn: For a square matrix A, the (classical) **adjoint** of A is the matrix

$$Adj\,A = [c_{ij}], \text{ where } c_{ij} = (-1)^{i+j} det A_{ji}.$$

In other words, the ij^{th} entry of $Adj\,A$ is the ji^{th} cofactor of A.

Find the adjoint of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5 & 0 & 6 \end{bmatrix}$

A square

A <u>NOT</u> invertible

$\Updownarrow$

$\det A = 0$

$\Updownarrow$

$\lambda = 0$ is an e. value of A

$\Updownarrow$

free variables

$\Updownarrow$

column linearly dependent

$\Updownarrow$ SQUARE

row of all zeros in EF

$\Updownarrow$

$\operatorname{col} A \neq \mathbb{R}^n$

$\Updownarrow$

etc

$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

e.g.

Ex: $\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$

What if A is <u>NOT</u> square

Ex $\begin{bmatrix} 1 & 0 & \boxed{\begin{matrix}3\\4\end{matrix}} \end{bmatrix} \sim \begin{bmatrix} 1 & \boxed{\begin{matrix}0\\0\end{matrix}} & \boxed{\begin{matrix}3\\0\end{matrix}} \\ 0 \end{bmatrix}$

or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \boxed{\begin{matrix}*\\0\\0\end{matrix}} \\ 0 & \\ 0 & \end{bmatrix}$

$\text{Col}A \neq \mathbb{R}^3$

$A\vec{x} = \vec{b}$ has unique sol'n

or no sol'n

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1. (1 pt) Library/Rochester/setLinearAlgebra23QuadraticForms-/ur_la_23_5.pg

If $A = \begin{bmatrix} 2 & -6 & 9 \\ -6 & 6 & -5 \\ 9 & -5 & -7 \end{bmatrix}$ and $Q(x) = x \cdot Ax$, $= x^T A x$

Then $Q(x_1,x_2,x_3) = \underline{2}\, x_1^2 + \underline{6}\, x_2^2 + \underline{-7}\, x_3^2 + \underline{-12}\, x_1x_2 + \underline{18}\, x_1x_3 + \underline{-10}\, x_2x_3$.

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2. (1 pt) Library/Rochester/setLinearAlgebra23QuadraticForms-/ur_la_23_1.pg

Write the matrix of the quadratic form

$Q(x) = -9x_1^2 + 2x_2^2 - 1x_3^2 - 1x_1x_2 + 3x_1x_3 - 9x_2x_3$.

$A = \begin{bmatrix} -9 & -1/2 & 3/2 \\ -1/2 & 2 & -9/2 \\ 3/2 & -9/2 & -1 \end{bmatrix}$.

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3. (1 pt) Library/Rochester/setLinearAlgebra23QuadraticForms-/ur_la_23_4.pg

If $A = \begin{bmatrix} 2 & 8 \\ 8 & 3 \end{bmatrix}$ and $Q(x) = x \cdot Ax$,

Then $Q(e_1) = \_\_\_$ and $Q(e_2) = \_\_\_$.

Hint: $e_1 = (1, 0)$
$e_2 = (0, 1)$

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4. (1 pt) Library/Rochester/setLinearAlgebra23QuadraticForms-/ur_la_23_2.pg

Find the eigenvalues of the matrix

$$M = \begin{bmatrix} -55 & 5 \\ 5 & -55 \end{bmatrix}.$$

Enter the two eigenvalues, separated by a comma:

Classify the quadratic form $Q(x) = x^T Ax$: >0 & <0

- A. $Q(x)$ is indefinite
- B. $Q(x)$ is negative definite >0
- C. $Q(x)$ is positive definite <0
- D. $Q(x)$ is positive semidefinite ≥ 0
- E. $Q(x)$ is negative semidefinite ≤ 0

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5. (1 pt) Library/Rochester/setLinearAlgebra23QuadraticForms-/ur_la_23_3.pg

The matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

has three distinct eigenvalues, $\lambda_1 < \lambda_2 < \lambda_3$,

$\lambda_1 = \_\_\_$,
$\lambda_2 = \_\_\_$,
$\lambda_3 = \_\_\_$.

Classify the quadratic form $Q(x) = x^T Ax$:

- A. $Q(x)$ is positive definite
- B. $Q(x)$ is positive semidefinite
- C. $Q(x)$ is indefinite
- D. $Q(x)$ is negative definite
- E. $Q(x)$ is negative semidefinite

e_2

e_1

7.2)

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -6 & 9 \\ -6 & 6 & -5 \\ 9 & -5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2x_1 - 6x_2 + 9x_3 \\ -6x_1 + 6x_2 - 5x_3 \\ 9x_1 - 5x_2 - 7x_3 \end{bmatrix}$$

$$= 2x_1^2 - 6x_1x_2 + \boxed{9x_1x_3}$$
$$+6x_2^2 - 6x_1x_2 \qquad -5x_2x_3$$
$$-7x_3^2 \qquad + \boxed{9x_1x_3} - 5x_2x_3$$

$$2x_1^2 + 6x_2^2 - 7x_3^2 - 12x_1x_2 + 18x_1x_3$$
$$-10x_2x_3$$

7.2

$$[x_1\ x_2\ x_3]\begin{bmatrix}3 & 2 & 1\\ 2 & 4 & \boxed{6}\\ 1 & \boxed{6} & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = Q(x_1, x_2, x_3)$$

$$Q(x_1\ x_2\ x_3) = [x_1\ x_2\ x_3]\begin{bmatrix}3x_1 + 2x_2 + 1x_3\\ 2x_1 + 4x_2 + 6x_3\\ x_1 + 6x_2 + 5x_3\end{bmatrix}$$

$$\begin{aligned}= \ & 3x_1^2 + 2x_1x_2 + x_1x_3\\ & 4x_2^2 + 2x_1x_2 + 6x_2x_3\\ & 5x_3^2 + x_1x_3 + 6x_2x_3\end{aligned}$$

$$3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 + 2x_1x_3 + \boxed{12}x_2x_3$$

$$= Q(x_1, x_2, x_3)$$

$$Q(x_1\ x_2\ x_3) = 5x_1^2 + 6x_2^2 + 3x_3^2 + 7x_1x_2 + 8x_1x_3$$

$$= x^T \begin{pmatrix} 5 & 7/2 & \frac{8}{2} \\ 7/2 & 6 & 0/2 \\ \frac{8}{2} & \frac{0}{2} & 3 \end{pmatrix} x$$

$$x^T A x$$