Inner Product Example: Dot product on R^n.

Defn: $\Sigma_{k=1}^m a_k = a_1 + a_2 + \ldots + a_m$

Defn:
The **dot product** of $\mathbf{u} = (u_1, \ldots, u_m)$ & $\mathbf{v} = (v_1, \ldots, v_m)$ is

$$\boxed{\mathbf{u} \cdot \mathbf{v} = \Sigma_{k=1}^m u_k v_k.}$$

In words, $\mathbf{u} \cdot \mathbf{v}$ is the sum of the products of the corresponding components of $\mathbf{u}$ and $\mathbf{v}$.

Note that $\boxed{\mathbf{u} \cdot \mathbf{v} \text{ is a real number (not a vector).}}$

Examples:

$(1, 2, 3) \cdot (4, 5, 6) = (1)(4) + (2)(5) + 3(6)$

$= 4 + 10 + 18$

$= 32$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = (1)(-2) + (2)(1)$

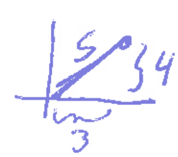
~~its~~ $= -2 + 2 = 0$

Defn: Let $\mathbf{v}$ be a vector in an inner product space $\mathbf{V}$. The **length** or **norm** of $\mathbf{v} = ||\mathbf{v}|| = \sqrt{< \mathbf{v}, \mathbf{v} >}$. $= \sqrt{v \cdot v}$

$||(3, 4)|| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$

Defn: The vector $\mathbf{u}$ is a **unit vector** if $||\mathbf{u}|| = 1$.

pythagorean theorem



6.1: Inner Products.

Defn: Let V be a vector space over the real numbers. An **inner product** for V is a function that associates a real number $\mathbf{u} \cdot \mathbf{v}$ to every pair of vectors, $\mathbf{u}$ and $\mathbf{v}$ in V such that the following properties are satisfied for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and scalars c:

a.) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b.) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

c.) $(c\mathbf{u}) \cdot v = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

related to length

d.) $\boxed{\mathbf{u} \cdot \mathbf{u} \geq 0 \text{ and } \mathbf{u} \cdot \mathbf{u} = 0 \text{ if and only if } \mathbf{u} = \mathbf{0}}$

A vector space V together with an inner product is called an **inner product space.**

$\sum_{i=1}^{k} u_i^2$

Thm 6.1.1′: Let V be an inner product space. Then for all vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{v}$ in V and scalars c_1, c_2:

a.) $(c_1\mathbf{u_1} + c_2\mathbf{u_2}) \cdot \mathbf{v} = \mathbf{v} \cdot (c_1\mathbf{u_1} + c_2\mathbf{u_2})$
$= c_1(\mathbf{u_1} \cdot \mathbf{v}) + c_2(\mathbf{u_2} \cdot \mathbf{v})$

b.) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

Defn: $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** (or **perpendicular**) if

$$\boxed{\mathbf{u} \cdot \mathbf{v} = 0.}$$

Example: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 1(-2) + 2(1) = 0$

Thus $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is a set of orthogonal unit vectors.

Example: $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} =$

Thus $\left\{ \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \right\}$ is a set of orthogonal unit vectors.

Observation: $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} =$

Suppose $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ is a pair of orthogonal unit vectors. Then

$$\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} \end{bmatrix} =$$

Note that $\frac{\mathbf{v}}{||\mathbf{v}||}$ is a unit vector.

Create a unit vector in the direction of the vector (3, 4):

$||(3,4)|| = \sqrt{9+16} = \sqrt{25} = 5$

Create a unit vector in the direction of the vector (1, 2):

Create a unit vector in the direction of the vector (-2, 1):