Sec 3.3 Cramer's Rule, Adjoint, Inverses

Defn: Let $A_i(b)$ = the matrix derived from A by replacing the ith column of A with b.

E.g. $A = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A_1(b) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $A_2(b) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Cramer's Rule

Suppose $A\underset{\sim}{x} = b$ where A is an $n \times n$ matrix such that $\det A \neq 0$. Then $x_i = \dfrac{\det A_i(b)}{\det A}$.

Example: Solve the following using Cramer's rule:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 5 \\ 6 \end{bmatrix}}_{b}$$

Solution:

$$x_1 = \frac{\det A_1(b)}{\det A}$$

$$x_2 = \frac{\det A_2(b)}{\det A}$$

②

$$\det A = \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1(4) - (3)(2) = -2$$

$$\det A_1 = \det \begin{bmatrix} 5 & 2 \\ 6 & 4 \end{bmatrix} = 5(4) - 6(2) = 8$$

$$\det A_2 = \det \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix} = 1(6) - 3(5) = -9$$

$$x_1 = \frac{8}{-2} = -4, \quad x_2 = \frac{-9}{-2} = 9/2$$

check:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ 9/2 \end{bmatrix} = \begin{bmatrix} -4+9 \\ -12+18 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Suppose $Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 & a_{12} \\ a_{21}x_1 + a_{22}x_2 & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$

$$A I_1(x) = A_1(b)$$

$$\det(A I_1(x)) = \det A_1(b)$$

$$\det(A) \underbrace{\det(I_1(x))}_{x_1(1) - x_2(0)} = \det A_1(b)$$

$$\det(A)\, x_1 = \det A_1(b)$$

$$x_1 = \frac{\det A_1(b)}{\det(A)} \qquad (\det A \neq 0)$$

In general,

$$A\, I_j(x) = [Ae_1 \cdots Ae_{j-1}\ Ax\ Ae_{j+1} \cdots Ae_n] = A_j(b)$$

$$\det(A\, I_j(x)) = \det A_j(b)$$

$$x_j = \frac{\det A_j(b)}{\det A}$$

Example: Solve the following using Cramer's rule:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5 & 0 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

Solution: $x_1 = \dfrac{\det A_1(b)}{\det A}$, $x_2 = \dfrac{\det A_2(b)}{\det A}$, $x_3 = \dfrac{\det A_3(b)}{\det A}$

$$\det A = \left| \begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 4 & 10 & 0 & 4 & 10 \\ 5 & 0 & 6 & 5 & 0 \end{array} \right| = 1(10)(6) + 2(0)(5) + 3(4)(0) - 5(10)(3) - 0(0)(1) - 6(4)(2)$$

$$= 60 + 0 + 0 - 150 - 0 - 48 = 60 - 198 = -138$$

$$\det A_1(b) = \left|\begin{array}{ccc|cc} 0 & 2 & 3 & 0 & 2 \\ 0 & 10 & 0 & 0 & 10 \\ 0 & 0 & 6 & 0 & 0 \end{array}\right| = 0$$

$$\det A_2(b) = \left|\begin{array}{ccc|cc} 1 & 0 & 3 & 1 & 0 \\ 4 & 0 & 0 & 4 & 0 \\ 5 & 0 & 6 & 5 & 0 \end{array}\right| = 0$$

$$\det A_3(b) = \left|\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 4 & 10 & 0 & 4 & 10 \\ 5 & 0 & 0 & 5 & 0 \end{array}\right| = 0$$

$$x_1 = \frac{0}{-138} = 0, \quad x_2 = \frac{0}{-138} = 0, \quad x_3 = \frac{0}{-138} = 0$$

Example: Solve the following using Cramer's rule:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\det A = \left|\begin{array}{ccc|cc} 2 & 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & 2 & 1 & 1 & 2 \end{array}\right| = \begin{array}{l} 2(-1)(1) + (1)(-1)(1) \\ + (1)(1)(2) - (-1) \\ - (-4) - (1) \end{array}$$

$$= \cancel{-2} - 1 + \cancel{2} + 1 + 4 - 1$$

$$= 3$$

$$\det A_1 = \left|\begin{array}{ccc|cc} 3 & 1 & 1 & 3 & 1 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 2 & 1 & 0 & 2 \end{array}\right. = \begin{array}{l} -3+0+0 \\ -0-(-6)+0 \\ = -3+6 = 3 \end{array}$$

$$\det A_2 = \left|\begin{array}{ccc|cc} 2 & 3 & 1 & 2 & 3 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right. = \begin{array}{l} 0-3+0-0-0-3 \\ = -6 \end{array}$$

$$\det A_3 = \left|\begin{array}{ccc|cc} 2 & 1 & 3 & 2 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 2 & 0 & 1 & 2 \end{array}\right. = \begin{array}{l} 0+0+6-(-3) \\ -0-0 \\ = 6+3 = 9 \end{array}$$

$$x_1 = \frac{3}{3} = 1, \quad x_2 = \frac{-6}{3} = -2, \quad x_3 = \frac{9}{3} = 3$$

Check:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2-2+3 \\ 1+2-3 \\ 1-4+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Adjoints

Defn: For a square matrix A, the (classical) adjoint of A is the matrix $\operatorname{Adj} A = [c_{ij}]$, where $c_{ij} = (-1)^{i+j} \det A_{ji}$.

In other words, the ij^{th} entry of Adj A is the ji^{th} cofactor of A.

Example: Find the adjoint of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5 & 0 & 6 \end{bmatrix}$

- Need to find cof A.
- Need to find the minors c_{ij}.

$$c_{11} = (-1)^{1+1} \begin{vmatrix} 10 & 0 \\ 0 & 6 \end{vmatrix} = 10(6) - 0 = 60$$

$$c_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 0 \\ 5 & 6 \end{vmatrix} = -4(6) - 0 = -24$$

$$c_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 10 \\ 5 & 0 \end{vmatrix} = 4(0) - 5(10) = -50$$

$$c_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -2(6) - 0(3) = -12$$

$$c_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} = 1(6) - 5(3) = -9$$

$$c_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} = -(1(0) - 10) = 10$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 10 & 0 \end{vmatrix} = 2(0) - 3(10) = -30$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix} = -\left(1(0) - 4(3)\right) = 12$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 4 & 10 \end{vmatrix} = 1(10) - 4(2) = 2$$

$$\text{cof } A = \begin{bmatrix} 60 & -24 & -50 \\ -12 & -9 & 10 \\ -30 & 12 & 2 \end{bmatrix}$$

$$\text{Adj } A = (\text{cof } A)^T = \begin{bmatrix} 60 & -12 & -30 \\ -24 & -9 & 12 \\ -50 & 10 & 2 \end{bmatrix}$$

Inverses

Thm: Let A be a square matrix, with $\det A \neq 0$. Then A is invertible, and $A^{-1} = \frac{1}{\det A} \text{Adj } A$.

Proof: Let $x = j$th column of A^{-1}.

Then $Ax = e_j$.

By Cramer's rule, $x_i = \frac{\det(A_i(e_j))}{\det(A)}$ = the (i,j) entry of A^{-1}

Example: Find the inverse of $\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5 & 0 & 6 \end{bmatrix}}_{A}$

Solution: $A^{-1} = \dfrac{1}{\det A} \operatorname{Adj} A$, $\det A = -138$

$$\cancel{\det A}\ A^{-1} = \frac{1}{-138} \begin{bmatrix} 60 & -12 & -30 \\ -24 & -9 & 12 \\ -50 & 10 & 2 \end{bmatrix}$$

Inverse of a 2x2 matrix:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{\det A} \operatorname{Adj} A$

$\det A = ad - bc$

$\cancel{\operatorname{Adj}}$ Cof $A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$\operatorname{Adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$