2.1: Operations on Matrices

$A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$.

Defn: Two matrices A and B are equal, if they have the same dimension and $a_{ij} = b_{ij}$ for all $i = 1, ..., n$, $j = 1, ..., m$

row i

column j

Defn: $A + B = (a_{ij} + b_{ij})$.

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$

Defn: $cA = (ca_{ij})$.

$3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) \\ 3(3) & 3(4) \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

Defn: $-B = (-1)B$.

Defn: $A - B = A + (-B)$.

Defn: The zero matrix $= \mathit{0} = (a_{ij})$ where $a_{ij} = 0$ for all i, j.

$0_{1\times 1}$ $0_{2\times 2}$ $0_{2\times 3}$ $0_{3\times 2}$

Ex: $\underbrace{[0]}_{1\times 1}$, $\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{2\times 2}$, $\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2\times 3}$, $\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{3\times 2}$, ...

rows columns

$I = I_2 = I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

mult

Defn: The identity matrix $= I = (a_{ij})$ where $a_{ij} = 0$ for all $i \neq j$ and $a_{ii} = 1$ for all i and I is a square matrix

$IA = A = AI$

as long as mult makes sense

Ex: $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, ...

$I_3 = I_{3\times 3}$

$A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$.

Suppose A is an $m \times k$ matrix, B is an $k \times n$. $AB = C$ where

$$c_{ij} = row(i)\ of\ A \cdot column(j)\ of\ B$$
$$= a_{i1}b_{1j} + a_{i2}b_{2k} + ... + a_{ir}b_{rk}.$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} = \begin{bmatrix} (1)(5)+2(8) & 1(6)+2(9) & 1(7)+2(10) \\ (3)(5)+4(8) & 3(6)+4(9) & 3(7)+4(10) \\ 8 & 9 & 10 \\ 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \\ 8 & 9 & 10 \\ 5 & 6 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = DNE$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Mult identity matrix}} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\boxed{4 \times 2}$

$a_{11} = 1$

row column

$\boxed{a_{12} = 2}$

row column

$a_{31} = 0$

row column

$$\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$2 \times \underline{3} \quad \underline{4} \times 2$

for mult to make sense

$(\underline{m} \times \underline{k})(\underline{k} \times \underline{n})$

must be =

$\cancel{3}\ 3 \neq 4 \Rightarrow$ mult not defined

Matrix mult is NOT commutative

2.1 cont: Note

$$AB \neq BA$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} (1)(1)+(1)(1) & 1(1)+1(1) \\ (-1)(1)+(-1)(1) & -1(1)+(-1)(1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

It is also possible that $AB = AC$, but $B \neq C$.

In particular it is possible for $AB = 0$, but $A \neq 0$ AND $B \neq 0$

Defn: If A is a square $(n \times n)$ matrix, $A^0 = I$, $A^1 = A$, $A^k = AA...A$.

Ex: $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

(The transpose of the $m \times n$ matrix A) $= A^T = (a_{ji})$.

Ex: $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Transpose Properties:

a.) $(A^T)^T = A$

b.) $(A + B)^T = A^T + B^T$

c.) $(kA)^T = kA^T$

d.) $(AB)^T = B^T A^T$

$A = (a_{ij})$, $A^T = (a_{ij})^T = (a_{ji})$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A \quad B \quad = \quad A \quad C$$

But $B \neq C$

Matrix mult is NOT COMM $AB \neq BA$

& We don't have cancellation property

$AB = AC \not\Rightarrow B = C$

$AB = AC$ But $B \neq C$

Matrix mult is not alway defined eg $(2\times 3)(1\times 1)$

$3 \neq 1 \Rightarrow$ can't multipl

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T\right)^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$(A^T)^T = A$$

Thm 1 (Properties of matrix arithmetic) Let A, B, C [redacted] be matrices. Let a, b be scalars. Assuming that the following operations are defined, then

a.) $A + B = B + A$

b.) $A + (B + C) = (A + B) + C$

c.) $A + 0 = A$

$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ d.) $A + (-A) = 0$

e.) $A(BC) = (AB)C$

f.) $\boxed{AI = A, IB = B}$

g.) $A(B + C) = AB + AC$,
$(B + C)A = BA + CA$

h.) $a(B + C) = aB + aC$

i.) $(a + b)C = aC + bC$

j.) $(ab)C = a(bC)$

k.) $a(AB) = (aA)B = A(aB)$

l.) $1A = A$

Defn.) $-A = -1A$

Cor.) $A0 = 0, 0B = 0$

Cor.) $a0 = 0$

Matrix mult is linear:

$A(r_1 B + r_2 C) = A(r_1 B) + A(r_2 C) = r_1 AB + r_2 AC$

where r_1, r_2 are real #'s (scalars)

MISSING: $AB \neq BA$

A^{-1} may not exist

(& cancellation law

2.2:

Defn: A is invertible if there exists a matrix B such that $AB = BA = I$, and B is called the inverse of A. If the inverse of A does not exist, then A is said to be singular.

Note that if A is invertible, then A is a square matrix.

$$\underset{m\times k}{A}\ \underset{k\times n}{B} = \underset{k\times n}{B}\ \underset{m\times k}{A}$$

$$\Rightarrow n = m$$

Thm: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if and only if $ad - bc \neq 0$, in which case

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

ch 3
$ad-bc$
$=$ determin

Ex: The inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

since

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4/-2 & -2/-2 \\ -3/-2 & 1/-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \blacksquare$$

$ad - bc = (1)(4) - 3(2) = 4 - 6 = -2$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4/-2 & -2/-2 \\ -3/-2 & 1/-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a d - b c

$(1)(4) + 2(-3) = 4-6 = -2$

$3(-2) + 4(1) = -2$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Will work but not determinant (Ch3)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -4/2 & 2/2 \\ 3/2 & -1/2 \end{bmatrix}$$

$-ad + bc = -$ determinant

$(-4) + 6 = 2$

$6 - 4 = 2$

Thm: Let A be a square matrix. If there exists a square matrix B such that $AB = I$, then $BA = I$ and thus $B = A^{-1}$

$\Leftrightarrow$

Thm: If A is invertible, then its inverse is unique.

Proof: Suppose $AB = I$ and $CA = I$. Then, $B = IB = (CA)B = C(AB) = CI = C$.

$2^0 = 1$

Defn: $A^0 = I$, and if n is a positive integer $A^n = AA \cdots A$ and $A^{-n} = A^{-1}A^{-1} \cdots A^{-1}$.

Thm: If r, s integers, $A^r A^s = A^{r+s}$, $(A^r)^s = A^{rs}$

Thm: If A^{-1} and B^{-1} exist, then

i.) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

ii.) A^{-1} is invertible and $(A^{-1})^{-1} = A$ $\leftarrow$ obvious

$A A^{-1} = I$

iii.) A^r is invertible and $(A^r)^{-1} = (A^{-1})^r$ where r is any integer

iv.) For any nonzero scalar k, kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$

v.) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Find the inverse of $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$.

Long method:
$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} =$$

$$\begin{bmatrix} 2x_{11} + 3x_{21} + 4x_{31} & 2x_{12} + 3x_{22} + 4x_{32} & 2x_{13} + 3x_{23} + 4x_{33} \\ 4x_{11} + 5x_{21} + 6x_{31} & 4x_{12} + 5x_{22} + 6x_{32} & 4x_{13} + 5x_{23} + fx_{33} \\ 6x_{11} + 7x_{21} + 9x_{31} & 6x_{12} + 7x_{22} + 9x_{32} & 6x_{13} + 7x_{23} + 9x_{33} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So solve,
$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 0 \end{array}\right] \text{ for } x_{11}, x_{21}, x_{31}.$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 1 \\ 6 & 7 & 9 & 0 \end{array}\right] \text{ for } x_{12}, x_{22}, x_{32}.$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 1 \end{array}\right] \text{ for } x_{13}, x_{23}, x_{33}.$$

Or shorter method, solve
$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 6 & 7 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cc} \overbrace{\qquad\qquad}^{A} & \overbrace{\qquad\qquad}^{I} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 6 & 7 & 9 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow (R_2 - 2R_1 \to R_2, R_3 - 3R_1 \to R_3)$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -2 & -3 & -3 & 0 & 1 \end{array}\right]$$

$\downarrow (-R_2 \to R_2)$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & -2 & -3 & -3 & 0 & 1 \end{array}\right]$$

$\downarrow (R_3 + 2R_2 \to R_3)$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$\downarrow (R_1 - 4R_3 \to R_1, R_2 - 2R_3 \to R_2)$

$$\begin{bmatrix} 2 & 3 & 0 & -3 & 8 & -4 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix}$$

$\downarrow (R_1 - 3R_2 \to R_1)$

$$\begin{bmatrix} 2 & 0 & 0 & -3 & -1 & 2 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix} \xrightarrow[(\frac{1}{2}R_1 \to R_1)]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\begin{array}{cc} \underbrace{\qquad\qquad}_{I} & \underbrace{\qquad\qquad\qquad}_{A^{-1}} \end{array}$$

A is invertible $\Leftrightarrow$ $A \sim I$

A square

Thus $\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 0 \end{array}\right]$ is row equivalent to $\begin{array}{c} \begin{array}{cccc} X_{11} & X_{21} & X_{31} & \phantom{-\frac{3}{2}} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}$,

so $(x_{11}, x_{21}, x_{31}) = (-\frac{3}{2}, 0, 1)$.

$\begin{bmatrix} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 1 \\ 6 & 7 & 9 & 0 \end{bmatrix}$ is row equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$,

so $(x_{12}, x_{22}, x_{32}) = (\frac{-1}{2}, 3, -2)$.

$\begin{bmatrix} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 1 \end{bmatrix}$ is row equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$,

so $(x_{13}, x_{23}, x_{33}) = (1, -2, 1)$.

When is A invertibl ??

Shortest method:

Note that if $[A|I]$ is row equivalent to $[I|B]$, then $B = A^{-1}$.

Thus the inverse of $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$ is $\begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Check answer: $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Elementary matrices and linear systems.

Definition: A matrix is called an elementary matrix if it can be obtained from an identity matrix by exactly one elementary row operation.

Examples:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow kR_1]{} \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 + kR_1 \leftrightarrow R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

elem row op $\Longleftrightarrow$ mult by an elem matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & j \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & j \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \xrightarrow{R_1 \leftrightarrow kR_1} \begin{bmatrix} ka & kb & kc \\ d & e & f \\ g & h & j \end{bmatrix}$$

$\times k \Rightarrow$ $$\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{bmatrix}$$ $\leftarrow \times k$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \xrightarrow{R_2 + kR_1 \leftrightarrow R_2} \begin{bmatrix} a & b & c \\ d+ka & e+kb & f+kc \\ g & h & j \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = \begin{bmatrix} a & b & c \\ ka+d & kb+e & \end{bmatrix}$$