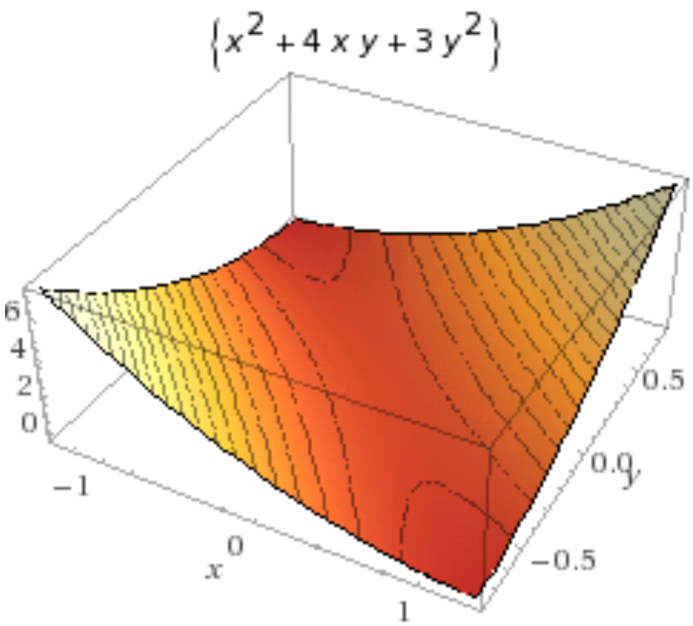
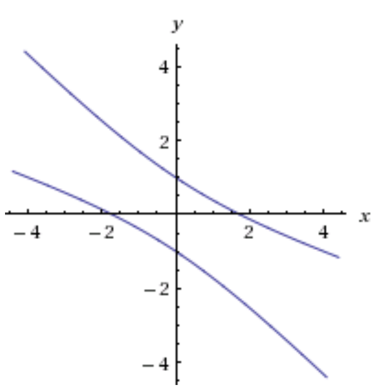
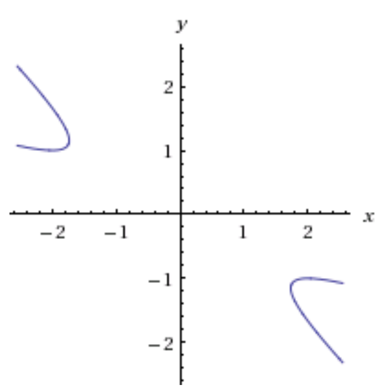
7.2: Quadratic Forms $\quad Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ where A is symmetric.

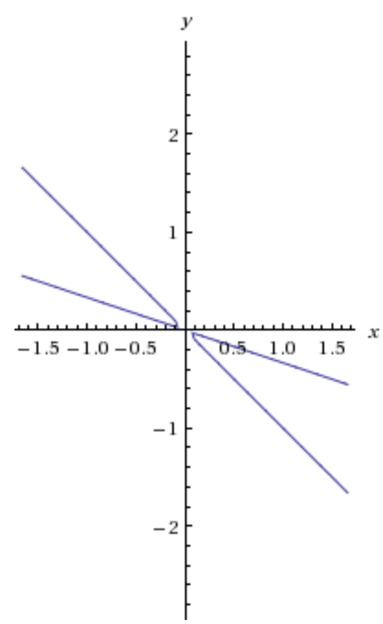
Example: $Q : R^2 \to R$

$$Q(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$





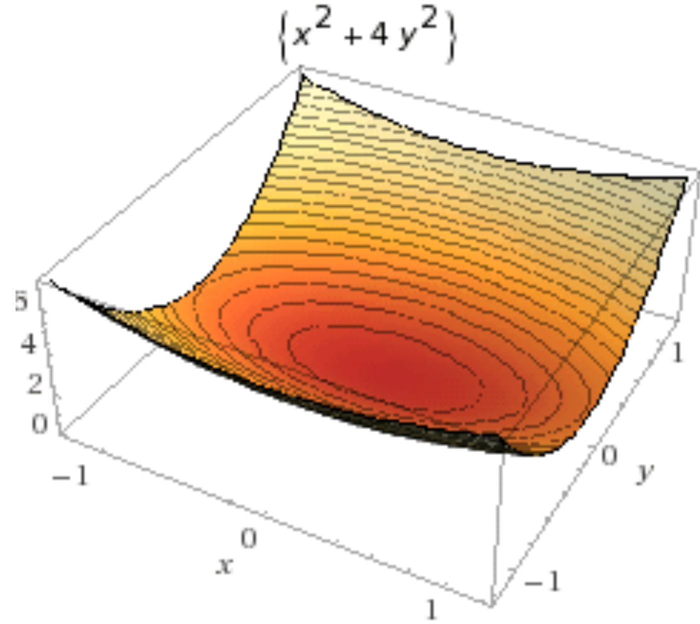


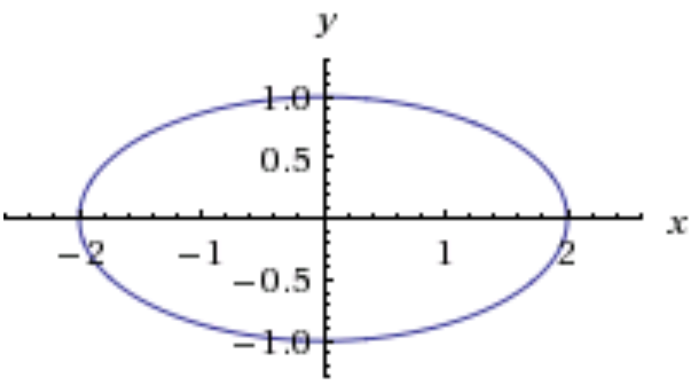


$x^2 + 4xy + 3y^2 = 4 \quad x^2 + 4xy + 3y^2 = -1 \quad x^2 + 4xy + 3y^2 = 0$

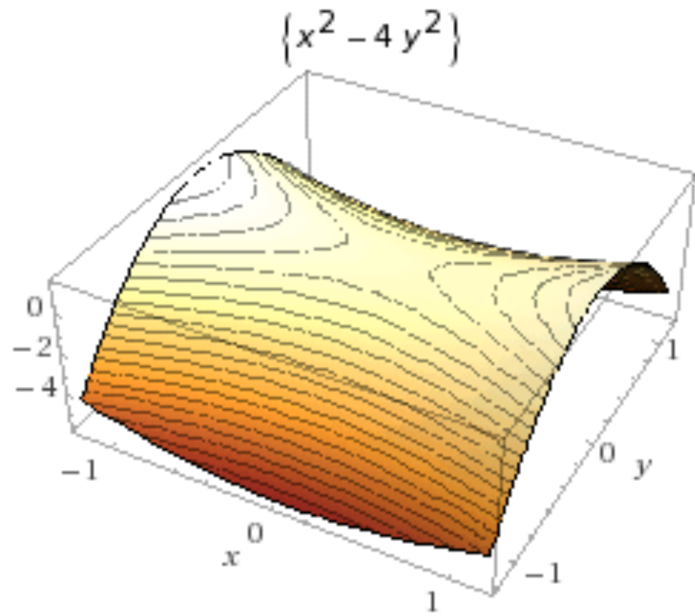
More examples: $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ where A is symmetric.

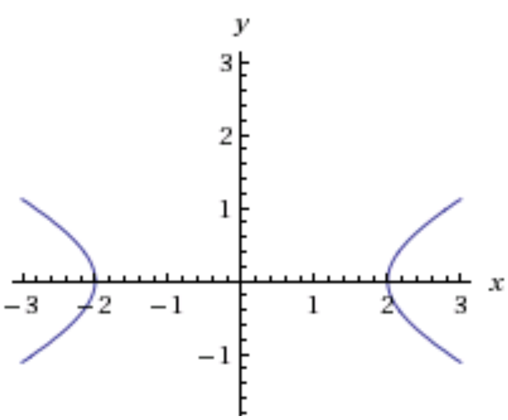
$$Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

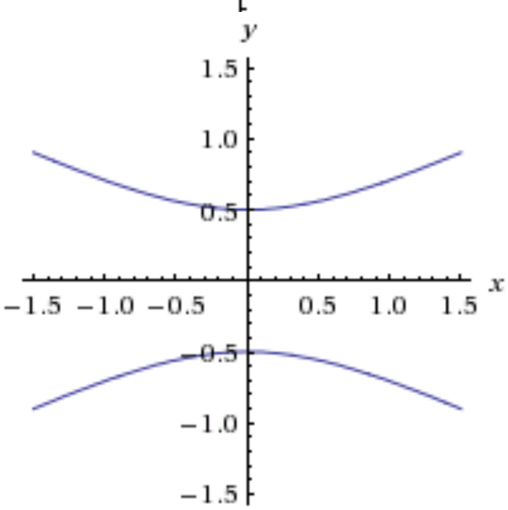


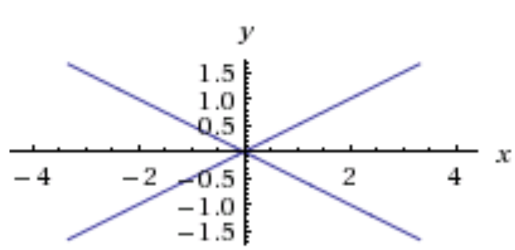


$$Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

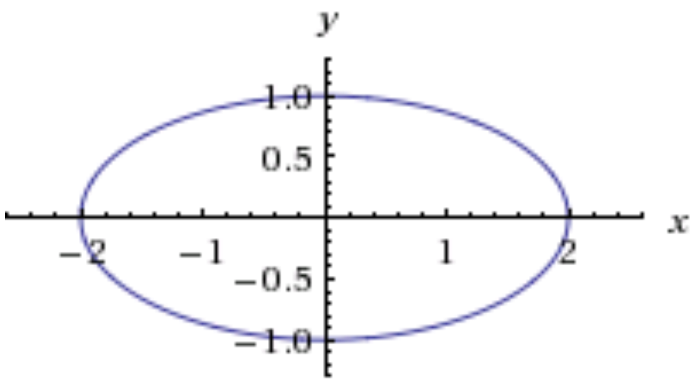


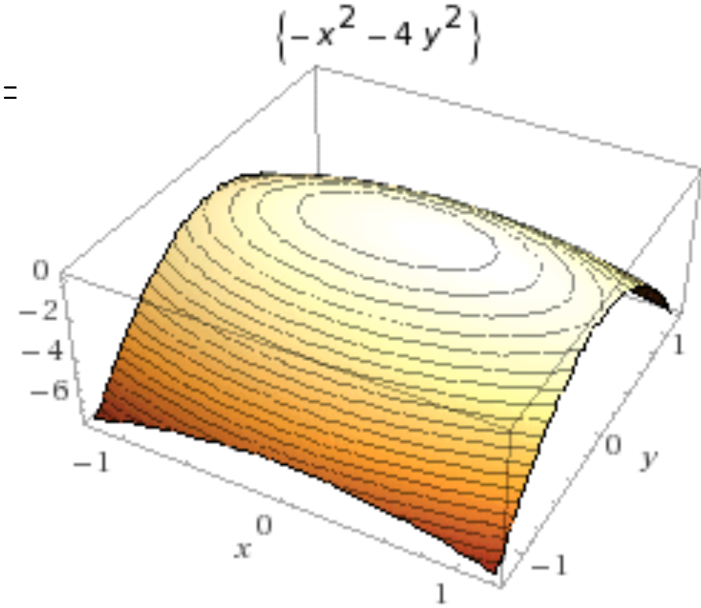




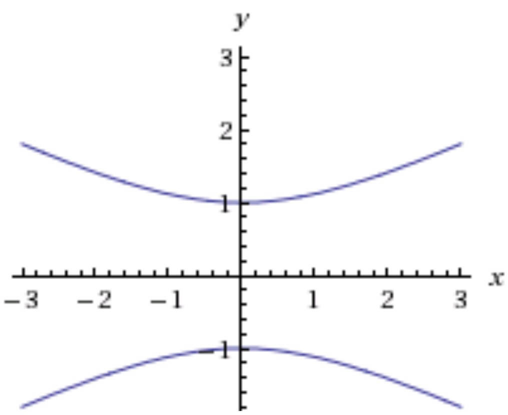


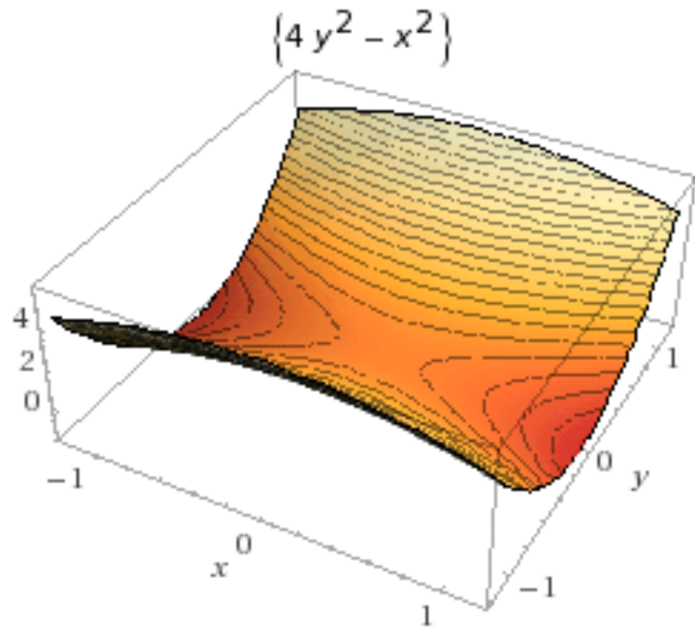
$$Q(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

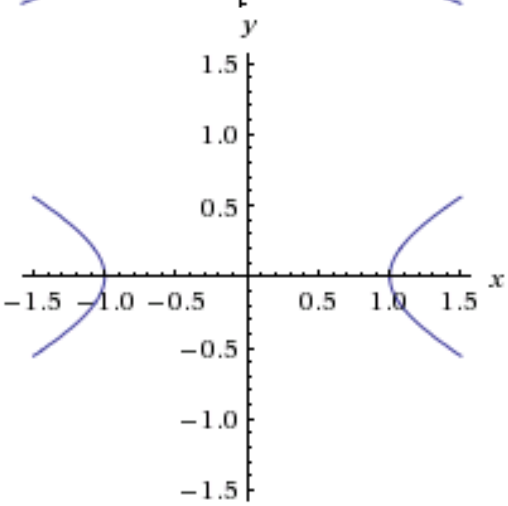


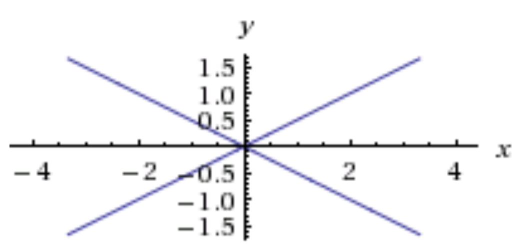


$$Q(x,y) = [x \quad y] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

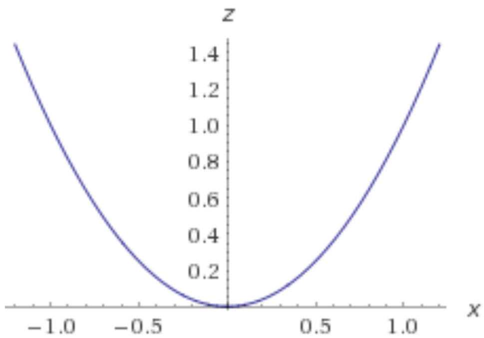




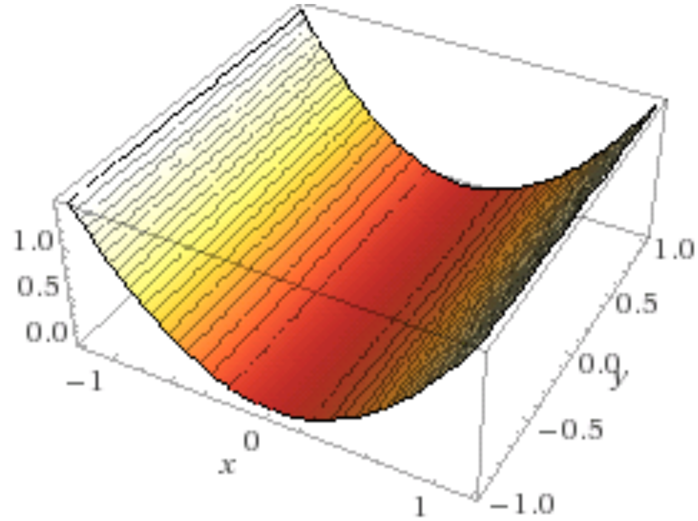




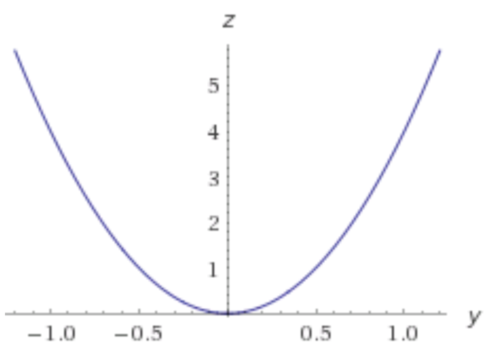
$$Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



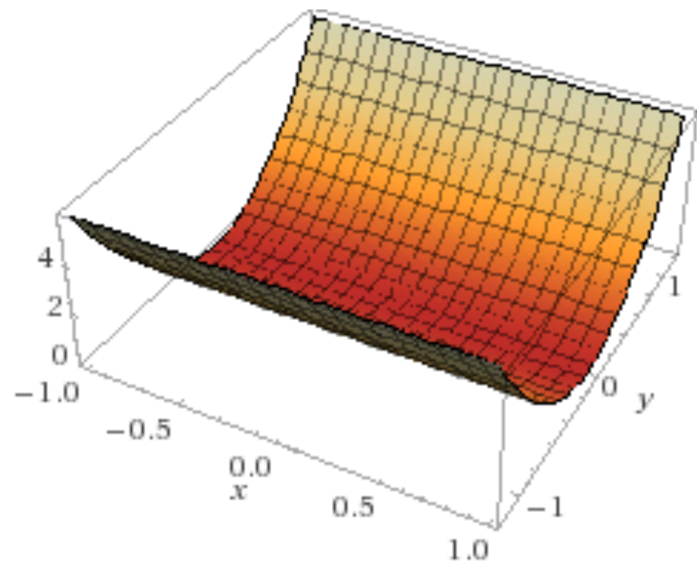
(x from −1 to 1)



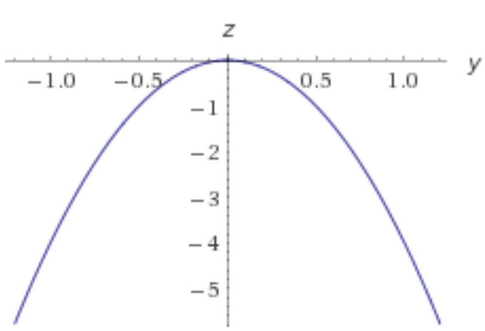
$$Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



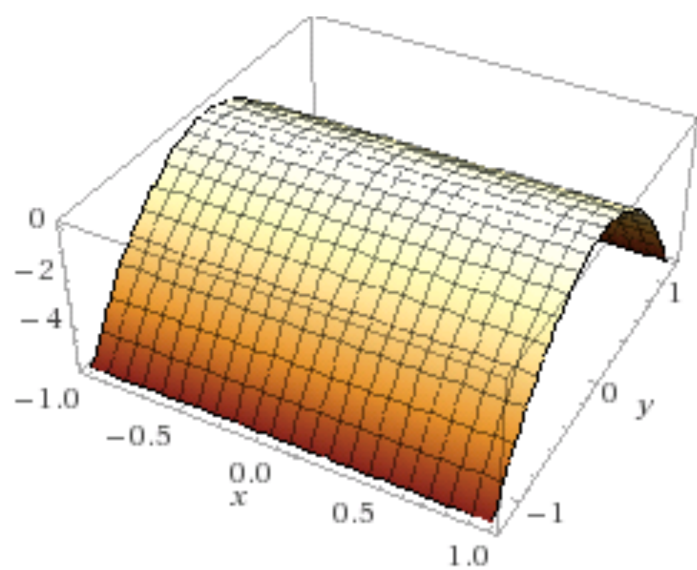
(y from −1 to 1)



$$Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



(y from −1 to 1)



Defn and theorem:

A symmetric matrix A is **positive definite**

if and only if the $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$

if and only if all the eigenvalues of A are positive.

A symmetric matrix A is **negative definite**

if and only if the $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$

if and only if all the eigenvalues of A are negative.

A symmetric matrix A is **indefinite**

if and only if the $\mathbf{x}^T A\mathbf{x}$ has both positive and negative values.

if and only if A are positive and negative eigenvalues.

A symmetric matrix A is **positive semidefinite**

if and only if the $\mathbf{x}^T A\mathbf{x} \geq 0$

if and only if all the eigenvalues of A are non-negative.

A symmetric matrix A is **negative semidefinite**

if and only if the $\mathbf{x}^T A\mathbf{x} \leq 0$

if and only if all the eigenvalues of A are non-positive.

Change of variable:

Let $\mathbf{x} = P\mathbf{y}$.

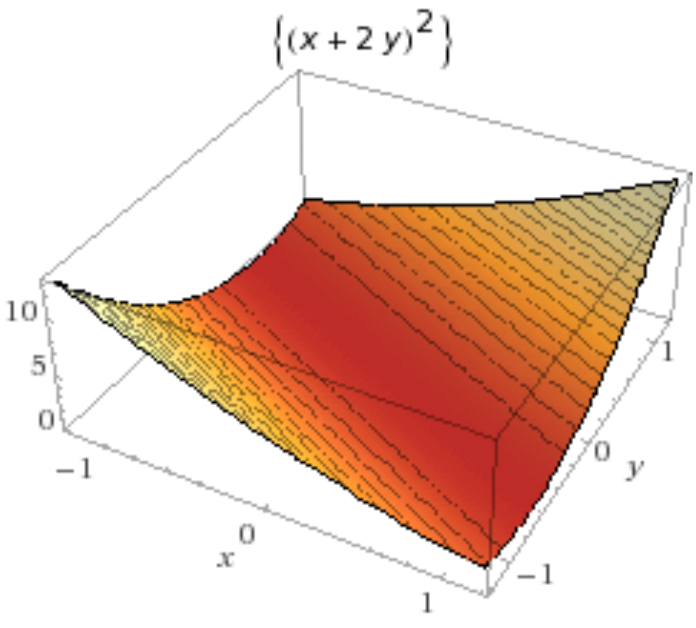
$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T AP\mathbf{y} = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T(P^T AP)\mathbf{y}$$

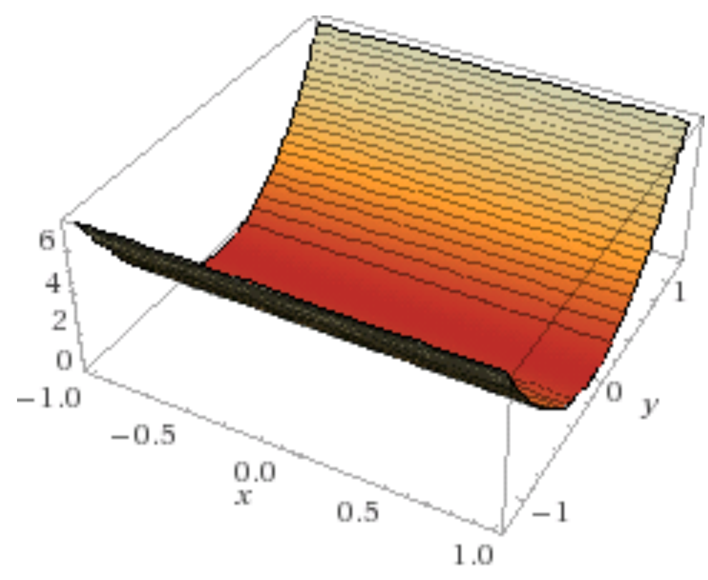
Suppose $A = PDP^{-1} = PDP^T$ where A is a symmetric matrix, D is diagonal, and P is orthonormal (i.e., $P^{-1} = P^T$).

$A = PDP^T$ implies $P^T AP = P^T PDP^T P = D$

$$Q(\mathbf{y}) = \mathbf{y}^T(P^T AP)\mathbf{y} = \mathbf{y}^T D\mathbf{y}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$





$$Q(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad Q(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Make a change of variable that transforms the following quadratic form into a quadratic form with no cross-product term:

$$Q(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Step 1: Orthogonally diagonalize $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

See section 7.1:

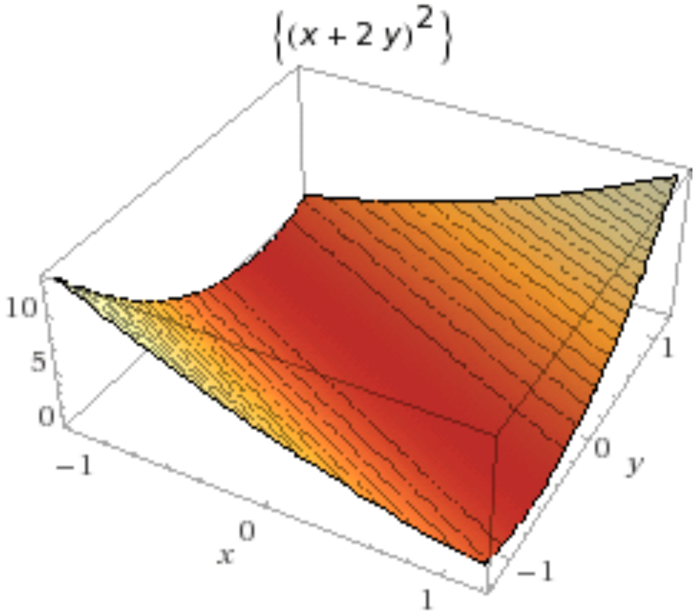
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = A = PDP^T = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

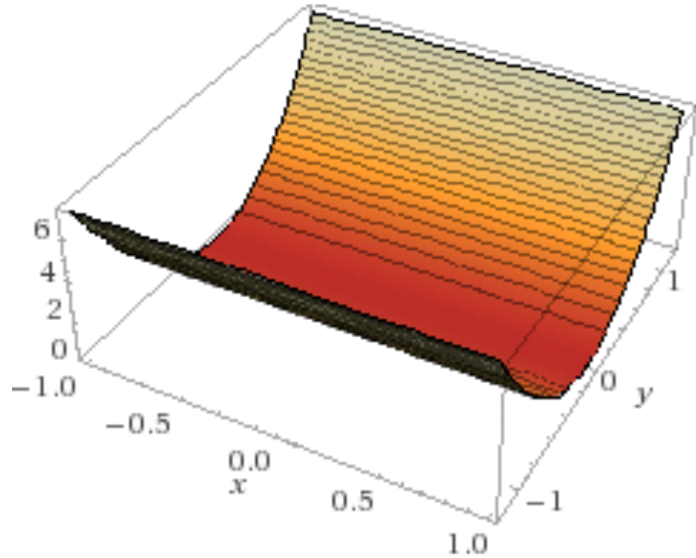
Step 2: Let $\mathbf{x} = P\mathbf{y}$

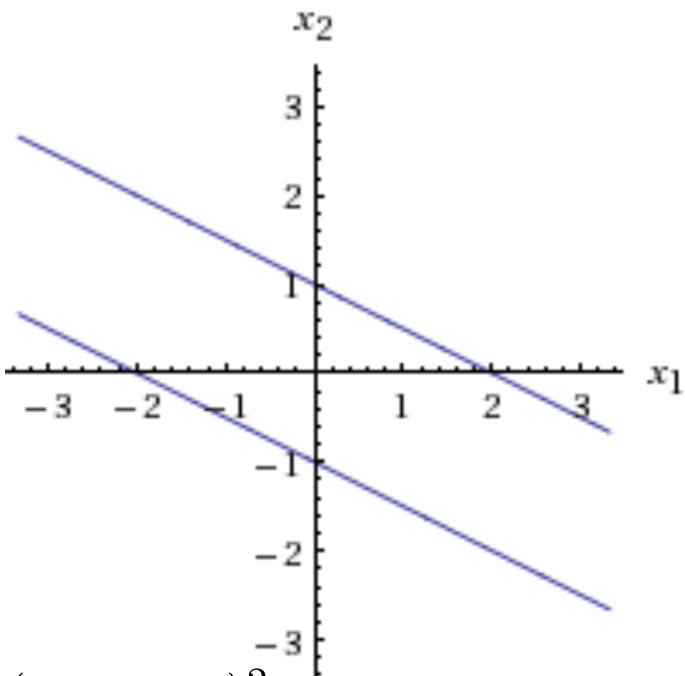
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} y_1 + \frac{1}{\sqrt{5}} y_2 \\ \frac{1}{\sqrt{5}} y_1 + \frac{2}{\sqrt{5}} y_2 \end{bmatrix}$$

After change of variable:

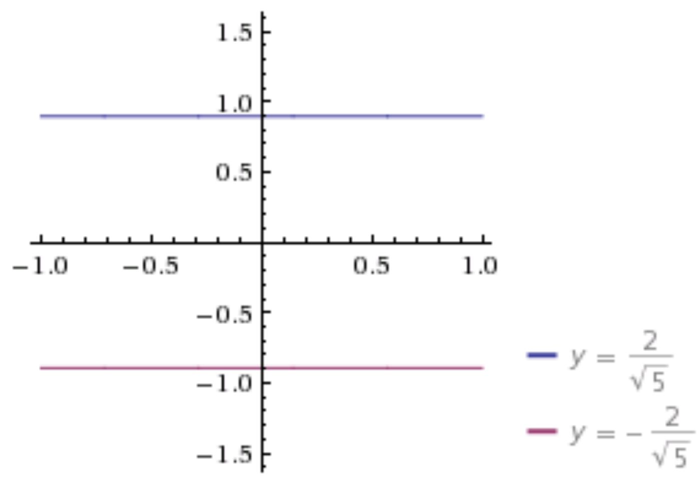
$$Q(y_1, y_2) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$







$$(x_1 + 2x_2)^2 = 4$$



$$((-\frac{2y_1}{\sqrt{5}} + \frac{y_2}{\sqrt{5}}) + 2(\frac{y_1}{\sqrt{5}} + \frac{2y_2}{\sqrt{5}}))^2 = 4$$