Suppose $A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $A\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

Claim:

If $A = A^T$ and $\lambda_1 \neq \lambda_2$, then $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is perpendicular to $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

I.e., If eigenvectors come from different eigenspaces, then the eigenvectors are orthogonal WHEN $A = A^T$.

Pf of claim: $\lambda_1(v_1, v_2) \cdot (w_1, w_2) = \lambda_1[v_1, v_2] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$$= (\lambda_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix})^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = (A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix})^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T A^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = [v_1, v_2] A \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= [v_1, v_2]\lambda_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda_2[v_1, v_2] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \lambda_2(v_1, v_2) \cdot (w_1, w_2)$$

$\lambda_1(v_1, v_2) \cdot (w_1, w_2) = \lambda_2(v_1, v_2) \cdot (w_1, w_2)$

implies $\lambda_1(v_1, v_2) \cdot (w_1, w_2) - \lambda_2(v_1, v_2) \cdot (w_1, w_2) = 0.$

Thus $(\lambda_1 - \lambda_2)(v_1, v_2) \cdot (w_1, w_2) = 0$

$\lambda_1 \neq \lambda_2$ implies $(v_1, v_2) \cdot (w_1, w_2) = 0$

Thus these eigenvectors are orthogonal.

7.1: Orthogonal Diagonalization

Equivalent Questions:

• Given an $n \times n$ matrix, does there exist an *orthonormal* basis for R^n consisting of *eigenvectors* of A?

• Given an $n \times n$ matrix, does there exist an orthonormal matrix P such that $P^{-1}AP = P^TAP$ is a diagonal matrix?

• Is A symmetric?

Defn: A matrix is **symmetric** if $A = A^T$.

Recall An invertible matrix P is **orthogonal** if $P^{-1} = P^T$

Defn: A matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.

Thm: If A is an $n \times n$ matrix, then the following are equivalent:

a.) A is orthogonally diagonalizable.

b.) There exists an orthonormal basis for R^n consisting of eigenvectors of A.

c.) A is symmetric.

Thm: If A is a symmetric matrix, then:

a.) The eigenvalues of A are all real numbers.

b.) Eigenvectors from different eigenspaces are orthogonal.

c.) Geometric multiplicity of an eigenvalue = its algebraic multiplicity

Note if $\{\mathbf{v_1}, ..., \mathbf{v_n}\}$ are linearly independent:

(1.) You can use the Gram-Schmidt algorithm to find an orthogonal basis for $span\{\mathbf{v_1}, ..., \mathbf{v_n}\}$. to pivot columns).

(2.) You can normalize these orthogonal vectors to create an orthonormal basis for $span\{\mathbf{v_1}, ..., \mathbf{v_n}\}$.

(3.) These basis vectors are not normally eigenvectors of $A = [\mathbf{v_1}...\mathbf{v_n}]$ even if A is symmetric (note that there are an infinite number of orthogonal basis for $span\{\mathbf{v_1}, ..., \mathbf{v_n}\}$ even if $n = 2$ and $span\{\mathbf{v_1}, \mathbf{v_2}\}$is just a 2-dimensional plane)

Note if A is a $n \times n$ square matrix that is diagonalizable, then you can find n linearly independent eigenvectors of A.

Each eigenvector is in $col(A)$: If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then $A\mathbf{v} = \lambda\mathbf{v}$. Thus $\frac{1}{\lambda}A\mathbf{v} = \mathbf{v}$.
Hence $A(\frac{1}{\lambda}\mathbf{v}) = \mathbf{v}$. Thus $\mathbf{v}$ is in $col(A)$.

Thus $col(A)$ is an $n-$dimensional subspace of R^n. That is $col(A) = R^n$, and you can find a basis for $col(A) = R^n$ consisting of eigenvectors of A.

But these eigenvectors are NOT usually orthogonal UNLESS they come from different eigenspaces AND the matrix A is symmetric.

If A is NOT symmetric, then eigenvectors from different eigenspaces need NOT be orthogonal.

IF A is symmetric,

To orthogonally diagonalize a symmetric matrix A:

1.) Find the eigenvalues of A.

$$\text{Solve } det(A - \lambda I) = 0 \text{ for } \lambda.$$

2.) Find a basis for each of the eigenspaces.

$$\text{Solve } (A - \lambda_j I)\mathbf{x} = 0 \text{ for } \mathbf{x}.$$

3.) Use the Gram-Schmidt process to find an orthonormal basis for each eigenspace.

That is for each λ_j use Gram-Schmidt to find an orthonormal basis for $Nul(A - \lambda_j I)$.

Eigenvectors from different eigenspaces will be orthogonal, so you don't need to apply Gram-Schmidt to eigenvectors from different eigenspaces

4.) Use the eigenvalues of A to construct the diagonal matrix D, and use the orthonormal basis of the corresponding eigenspaces for the corresponding columns of P.

5.) Note $P^{-1} = P^T$ since the columns of P are orthonormal.

Example 1:

Orthogonally diagonalize $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Step 1: Find the eigenvalues of A:

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4$$

$$= \lambda^2 - 5\lambda + 4 - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$$

Thus $\lambda = 0, 5$ are are eigenvalues of A.

2.) Find a basis for each of the eigenspaces:

$$\lambda = 0 : (A - 0I) = A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Thus $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 0.

$$\lambda = 0 : (A - 5I) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

Thus $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A with eigenvalue 5.

3.) Create orthonormal basis:

Since A is symmetric and the eigenvectors $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ come from different eigenspaces (ie their eigenvalues are different), these eigenvectors are orthogonal. Thus we only need to normalize them:

$$\left|\left| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right|\right| = \sqrt{4+1} = \sqrt{5}$$

$$\left|\left| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right|\right| = \sqrt{1+4} = \sqrt{5}$$

Thus an orthonormal basis for $col(A) = R^2 = \left\{ \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \right\}$

4.) Construct D and P

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Make sure order of eigenvectors in D match order of eigenvalues in P.

5.) P orthonormal implies $P^{-1} = P^T$

Thus $P^{-1} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

Note that in this example, $P^{-1} = P$, but that is NOT normally the case.

Thus $A = P^{-1}DP$

Thus $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

Example 2:
Orthogonally diagonalize $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Step 1: Find the eigenvalues of A:

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 0 & -\lambda & -\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ -\lambda & -\lambda \end{vmatrix} - (-1) \begin{vmatrix} -1 & 1 \\ -\lambda & -\lambda \end{vmatrix} + 0 \begin{vmatrix} -1 & 1 \\ 1-\lambda & -1 \end{vmatrix}$$

$= (1-\lambda)[(1-\lambda)(-\lambda) - \lambda] + [\lambda + \lambda]$
$= (1-\lambda)(-\lambda)[(1-\lambda) + 1] + 2\lambda = (1-\lambda)(-\lambda)(2-\lambda) + 2\lambda$

Note I can factor out $-\lambda$, leaving only a quadratic to factor:

$= -\lambda[(1-\lambda)(2-\lambda) - 2]$
$= -\lambda[\lambda^2 - 3\lambda + 2 - 2] = -\lambda[\lambda^2 - 3\lambda] = -\lambda^2[\lambda - 3]$

Thus their are 2 eigenvalues:

$\lambda = 0$ with algebraic multiplicity 2. Since A is symmetric, geometric multiplicity = algebraic multiplicity = 2. Thus the dimension of the eigenspace corresponding to $\lambda = 0$ [$= Nul(A - 0I) = Nul(A)$] is 2.

$\lambda = 3$ w/algebraic multiplicity = 1 = geometric multiplicity.

Thus we can find an orthogonal basis for R^3 where two of the basis vectors comes from the eigenspace corresponding to eigenvalue 0 while the third comes from the eigenspace corresponding to eigenvalue 3.

2.) Find a basis for each of the eigenspaces:

2a.) $\lambda = 0 : A{-}0I = A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus a basis for eigenspace corresponding to eigenvalue 0 is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

We can now use Gram-Schmidt to turn this basis into an orthogonal basis for the eigenspace corresponding to eigenvalue 0 or we can continue finding eigenvalues.

3a.) Create orthonormal basis using Gram-Schmidt for the eigenspace corresponding to eigenvalue 0:

Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$proj_{\mathbf{v_1}} \mathbf{v_2} = \frac{\mathbf{v_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} = \frac{-1+0+0}{1+1+0} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

The vector component of $\mathbf{v_2}$ orthogonal to $\mathbf{v_1}$ is

$$\mathbf{v_2} - proj_{\mathbf{v_1}}\mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Thus an orthogonal basis for the eigenspace corresponding to eigenvalue 0 is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

To create orthonormal basis, divide each vector by its length:

$$\left\| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

$$\left\| \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}$$

Thus an orthonormal basis for the eigenspace corresponding to eigenvalue 0 is

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2\sqrt{3}} \\ \frac{\sqrt{2}}{2\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \right\}$$

2b.) Find a basis for eigenspace corresponding to $\lambda = 3$:

$$A-3I = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \blacksquare$$

Thus a basis for eigenspace corresponding to eigenvalue 3 is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

FYI: Alternate method to find 3rd vector: Since you have two linearly independent vectors from the eigenspace corresponding to eigenvalue 0, you only need one more vector which is orthogonal to these two to form a basis for R^3. Note since A is symmetric, any such vector will be an eigenvector of A with eigenvalue 3. Note this shortcut only works because we know what the eigenspace corresponding to eigenvalue 3 looks like: a line perpendicular to the plane representing the eigenspace corresponding to eigenvalue 0.

3b.) Create orthonormal basis for the eigenspace corresponding to eigenvalue 3:

We only need to normalize:

$$\left\| \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

Thus orthonormal basis for eigenspace corresponding to eigenvalue 3 is

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\}$$

4.) Construct D and P

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Make sure order of eigenvectors in D match order of eigenvalues in P.

5.) P orthonormal implies $P^{-1} = P^T$

Thus $P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Thus $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = A = P^{-1}DP =$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$