Orthonormal Bases.

A set of vectors, $\mathcal{S}$, is an **orthogonal set** if every pair of distinct vectors is orthogonal.

A set $\mathcal{T}$, is an **orthonormal set** if it is an orthogonal set and if every vector in $\mathcal{T}$ has norm equal to 1.

Thm: Let $\mathcal{T} = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be an orthogonal set of nonzero vectors in an inner product space V. Then $\mathcal{T}$ is linearly independent.

Cor: An orthonormal set of vectors is linearly independent.

Defn: Let $\mathbf{V}$ be an inner product space. If $V = span\mathcal{T}$ &

i.) if $\mathcal{T}$ is an orthogonal set, then $\mathcal{T}$ is an **orthogonal basis** for V.

ii.) if $\mathcal{T}$ is orthonormal set, then $\mathcal{T}$ is an **orthonormal basis** for V.

Thm: Let $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be an orthogonal basis for an inner product space V. Let $\mathbf{a}$ be an arbitrary vector in V. Then

$$\mathbf{a} = c_1\mathbf{v_1} + ... + c_n\mathbf{v_n}$$

$$\text{where } c_j = \frac{<\mathbf{a}, \mathbf{v_j}>}{||\mathbf{v_j}||^2} \text{ for } j = 1, 2, ..., n.$$

Note if $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is an orthonormal basis, then

$$||\mathbf{v_j}|| = 1 \text{ and } c_j =< \mathbf{a}, \mathbf{v_j} >$$

Thm: Let $\mathbf{a}$, $\mathbf{v}$ be nonzero vectors in R^k.

The vector component of $\mathbf{a}$ along $\mathbf{v}$

$= \text{orthogonal projection of } \mathbf{a} \text{ on } \mathbf{v}$

$= proj_{\mathbf{v}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{v}}{||\mathbf{v}||^2}\mathbf{v}$

The vector component of $\mathbf{a}$ orthogonal to $\mathbf{v}$

$= \mathbf{a} - proj_{\mathbf{v}}\mathbf{a} = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{v}}{||\mathbf{v}||^2}\mathbf{v}$

Thm: Let $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be an orthogonal basis for subspace W of an inner product space V. Let $\mathbf{a}$ be an arbitrary vector in V. Then

$$proj_W\mathbf{a} = c_1\mathbf{v_1} + ... + c_n\mathbf{v_n}$$

$$\text{where } c_j = \frac{<\mathbf{a}, \mathbf{v_j}>}{||\mathbf{v_j}||^2} \text{ for } j = 1, 2, ..., n.$$

Note if $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is an orthonormal basis, then

$$||\mathbf{v_j}|| = 1 \text{ and } c_j =< \mathbf{a}, \mathbf{v_j} >$$

The vector component of $\mathbf{a}$ orthogonal to $\mathbf{W} = \mathbf{a} - proj_{\mathbf{W}}\mathbf{a}$

Thm (Gram-Schmidt process for constructing an orthogonal basis):

Let $\mathcal{T} = \{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}\}$ be a basis for an inner product space V. Let $\mathcal{T}' = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be defined as follows:

$$\mathbf{v_1} = \mathbf{a_1}$$

$$\mathbf{v_2} = \mathbf{a_2} - \frac{<\mathbf{a_2}, \mathbf{v_1}>}{<\mathbf{v_1}, \mathbf{v_1}>}\mathbf{v_1}$$

$$\mathbf{v_3} = \mathbf{a_3} - \frac{<\mathbf{a_3}, \mathbf{v_1}>}{<\mathbf{v_1}, \mathbf{v_1}>}\mathbf{v_1} - \frac{<\mathbf{a_3}, \mathbf{v_2}>}{<\mathbf{v_2}, \mathbf{v_2}>}\mathbf{v_2}$$

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$$\mathbf{v_n} = \mathbf{a_n} - \frac{<\mathbf{a_n}, \mathbf{v_1}>}{<\mathbf{v_1}, \mathbf{v_1}>}\mathbf{v_1} - \frac{<\mathbf{a_n}, \mathbf{v_2}>}{<\mathbf{v_2}, \mathbf{v_2}>}\mathbf{v_2} - ... - \frac{<\mathbf{a_n}, \mathbf{v_n}>}{<\mathbf{v_n}, \mathbf{v_n}>}\mathbf{v_n}$$

Then the set $\mathcal{T}'$ is an orthogonal basis for V.

An orthonormal basis for V is given by

$$\mathcal{T}'' = \left\{ \frac{\mathbf{v_1}}{||\mathbf{v_1}||}, \frac{\mathbf{v_2}}{||\mathbf{v_2}||}, ..., \frac{\mathbf{v_n}}{||\mathbf{v_n}||} \right\}.$$

The QR Decomposition.

Note: If the columns of Q are orthonormal, then $Q^TQ = I$.

$$Q^TQ = \begin{bmatrix} q_1^T \\ q_2^T \\ \cdot \\ \cdot \\ \cdot \\ q_n^T \end{bmatrix} [q_1 \; q_2 ... q_n] = \begin{bmatrix} q_1 \cdot q_1 & q_1 \cdot q_2 & \cdots & q_1 \cdot q_n \\ q_2 \cdot q_1 & q_2 \cdot q_2 & \cdots & q_2 \cdot q_n \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ q_n \cdot q_1 & q_n \cdot q_2 & \cdots & q_n \cdot q_n \end{bmatrix}$$

Defn: $A = QR$ is a **QR factorization** of A if Q is has orthonormal columns and R is upper-triangular.

Finding a QR-factorization of A:

1.) Find Q: Apply the Gram-Schmidt algorithm to the columns of A to find an orthonormal basis, $\{q_1, ..., q_n\}$ for the column space of A.

$$Q = [q_1 \;\; q_2 ... q_n].$$

2.) Find R: $R = Q^TA$.

Thm: If A is an matrix with linearly independent column vectors, then there exists a QR-factorization of A.