5.1: Eigenvalues and Eigenvectors

Defn: λ is an **eigenvalue** of the matrix A if there exists a <u>nonzero</u> vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

The vector $\mathbf{x}$ is said to be an **eigenvector** corresponding to the eigenvalue λ.

Example: Let $A = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of A and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ for any k.

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is <u>NOT</u> an eigenvector of A.

MOTIVATION:

Note $\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus $A \begin{bmatrix} 2 \\ 8 \end{bmatrix} = A(\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = A \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \cdot 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$$

Finding eigenvalues:

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ (Note A is a SQUARE matrix).

Then $A\mathbf{x} = \lambda I\mathbf{x}$ where I is the identity matrix.

Thus $A\mathbf{x} - \lambda I\mathbf{x} = (A - \lambda I)\mathbf{x} = \mathbf{0}$

Thus if $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Thus $det(A - \lambda I)\mathbf{x} = 0$.

Note that the eigenvectors corresponding to λ are the nonzero solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Thus to find the eigenvalues of A and their corresponding eigenvectors:

Step 1: Find eigenvalues: Solve the equation

$$det(A - \lambda I) = 0 \text{ for } \lambda.$$

Step 2: For each eigenvalue λ_0, find its corresponding eigenvectors by solving the homogeneous system of equations

$$(A - \lambda_0 I)\mathbf{x} = 0 \text{ for } \mathbf{x}.$$

Defn: $det(A - \lambda I) = 0$ is the **characteristic equation** of A.

Thm 3: The eigenvalues of an upper triangular or lower triangular matrix (including diagonal matrices) are identical to its diagonal entries.

Defn: The **eigenspace** corresponding to an eigenvalue λ_0 of a matrix A is the set of all solutions of $(A - \lambda_0 I)\mathbf{x} = \mathbf{0}$.

Note: An eigenspace is a vector space

The vector $\mathbf{0}$ is always in the eigenspace.

The vector $\mathbf{0}$ is never an eigenvector.

The number 0 can be an eigenvalue.

Thm: A square matrix is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Thm: If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^k\mathbf{x} = \lambda^k\mathbf{x}$. That is, if λ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$, then λ^k is an eigenvalue of A^k with corresponding eigenvector $\mathbf{x}$ where k is any integer.

Defn: Suppose the characteristic polynomial of A is

$$(\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2}...(\lambda - \lambda_n)^{k_p}$$

where the λ_i, $i = 1, ..., p$ are DISTINCT. Then the **algebraic multiplicity of** λ_i is k_i.
That is the **algebraic multiplicity of** λ_i is the number of times that $(\lambda - \lambda_i)$ appears as a factor of the characteristic polynomial of A.

Defn: The **geometric multiplicity of** λ_i
= dimension of the eigenspace corresponding to λ_i.

Thm (Geometric and Algebraic Multiplicity): The geometric multiplicity is less than or equal to the algebraic multiplicity [That is, Nullity of $(A - \lambda_i I) \leq k_i$].

Find the eigenvalues and their corresponding eigenspace of
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Find the eigenvalues and their corresponding eigenspace of
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Find the eigenvalues and their corresponding eigenspace of
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$