Ch 2 partial review:

Recall W is a **subspace** of R^n (**vector space**) if W is closed under scalar multiplication and vector addition.

I.e., W is a subspace of R^n if

$$\mathbf{v_1}, \mathbf{v_2} \text{ in } W \text{ implies } c_1\mathbf{v_1} + c_2\mathbf{v_2} \text{ in } W.$$

Note if W is a finite dimensional subspace, then for some vectors $\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_k}$ in W:

$$W = span\{\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_k}\}$$

$$= \{c_1\mathbf{w_1} + c_2\mathbf{w_2} + ... + c_k\mathbf{w_k} \mid c_i \in R\}$$

$=$ the set of all linear combinations of the vectors $\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_k}$.

Examples:

The column space of $A = [\mathbf{a_1}\ \mathbf{a_2}\ ...\ \mathbf{a_n}]$

$$= \{c_1\mathbf{a_1} + c_2\mathbf{a_2} + ... + c_n\mathbf{a_n} \mid c_i \in R\}$$

$$= \{\mathbf{b} \mid\ A\mathbf{x} = \mathbf{b} \text{ has at least one solution } \}$$

is a subspace.

Nullspace of A = solution set of $A\mathbf{x} = \mathbf{0}$ is a subspace:

If $\mathbf{v_1}, \mathbf{v_2}$ are solutions to $A\mathbf{x} = \mathbf{0}$, then $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ is also a solution:

The solution set of $A\mathbf{x} = \mathbf{b}$ is NOT a subspace unless $\mathbf{b} = \mathbf{0}$:

If $\mathbf{v_1}, \mathbf{v_2}$ are solutions to $A\mathbf{x} = \mathbf{b}$, then $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ is a solution to

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Ch 5: The eigenspace corresponding to an eigenvalue λ is a subspace.

Defn: Let W be a subspace of R^k. A set $\mathcal{T}$ is a **basis** for W if

i.) $\mathcal{T}$ is linearly independent and
ii.) $\mathcal{T}$ spans W.

I.e.,
$\mathcal{T}$ is the smallest collections of vectors that span W.

Basis thm: Let W be a p-dimensional subspace of R^n.

i.) If $W = \text{span}\{w_1, ..., w_p\}$, then $\{w_1, ..., w_p\}$ is a basis for W.

ii.) If $v_1, ..., v_p$ are linearly independent vectors in W, then $\{v_1, ..., v_p\}$ is a basis for W.

Thm: All basis for a finite-dimensional vector space have the same number of elements.

Defn:
$dim(V)$ = the **dimension** of a finite-dim vector sp V
= the number of vectors in any basis for V.

If $dim(V) = n$, then V is said to be n-dimensional.

rank A = Rank of a matrix A = dimension of Col A
= number of pivot columns of A.

nullity of A = dimension of Nul A
= number of free variables.

Rank(A) + nullity(A) = Number of columns of A.

That is,

The number of pivots of A	+	The number of free variables of A	=	The number of columns of A

Ex. 1) Suppose A is a $5X7$ matrix.

If Rank(A) = 4, then nullity(A) =

$A\mathbf{x} = \mathbf{0}$ has ____________________ solutions.

$A\mathbf{x} = \mathbf{b}$ has ____________________ solutions.

If Rank(A) = 5, then nullity(A) =

$A\mathbf{x} = \mathbf{0}$ has ____________________ solutions.

$A\mathbf{x} = \mathbf{b}$ has ____________________ solutions.

If Rank(A) = 5, the column space of A =

3.3: Cramer's Rule, Adjoint, Inverses, Area

Defn: Let $A_i(\mathbf{b})$ = the matrix derived from A by replacing the i^{th} column of A with $\mathbf{b}$.

Cramer's Rule: Suppose $A\mathbf{x} = \mathbf{b}$ where A is an $n \times n$ matrix such that $det A \neq 0$. Then

$$x_i = \frac{det A_i(\mathbf{b})}{det A}.$$

Solve the following using Cramer's rule:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (3)(2) = 4 - 6 = -2$$

$$det \begin{bmatrix} 5 & 2 \\ 6 & 4 \end{bmatrix} = (5)(4) - (6)(2) = 20 - 12 = 8$$

$$det \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix} = (1)(6) - (3)(5) = 6 - 15 = -9$$

Thus $x_1 = \frac{8}{-2} = -4, \quad x_2 = \frac{-9}{-2} = \frac{9}{2}$.

Observe for 2×2 case:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 & a_{12} \\ a_{21}x_1 + a_{22}x_2 & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$

$$AI_1(\mathbf{x}) = A_1(\mathbf{b})$$

$$det(AI_1(\mathbf{x})) = det(A_1(\mathbf{b}))$$

$$det(A)\ det(I_1(\mathbf{x})) = det(A_1(\mathbf{b}))$$

$$det(A)\ x_1 = det(A_1(\mathbf{b}))$$

Thus $x_1 = \frac{det(A_1(\mathbf{b}))}{det(A)}$

$$AI_j(\mathbf{x}) = [A\mathbf{e_1}\ ...\ A\mathbf{e_{j-1}}\ A\mathbf{x}\ A\mathbf{e_{j+1}}\ ...\ \mathbf{Ae_n}] = A_j(\mathbf{b})$$

Solve the following using Cramer's rule:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Defn: For a square matrix A, the (classical) **adjoint** of A is the matrix

$$AdjA = [c_{ij}], \text{ where } c_{ij} = (-1)^{i+j} detA_{ji}.$$

In other words, the ij^{th} entry of $AdjA$ is the ji^{th} cofactor of A.

Find the adjoint of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5 & 0 & 6 \end{bmatrix}$

Thm: Let A be a square matrix, with $detA \neq 0$. Then A is invertible, and

$$A^{-1} = \frac{1}{detA} Adj A.$$

Proof:
Let $\mathbf{x} =$ the jth column of A^{-1}. Then $A\mathbf{x} = \mathbf{e_j}$

By Cramer's rule, $x_i = \frac{det(A_i(\mathbf{e_j}))}{det(A)}$

$=$ the (i, j) entry of A^{-1}

Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5 & 0 & 6 \end{bmatrix}$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $Adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$det A = ad - bc.$

Thus $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Area and Volume

a.) The area of the parallelogram in 2-space determined by the vectors (u_1, u_2) and (v_1, v_2)

$$= \left| det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right|$$

b.) The volume of the parallelepiped in 3-space determined by the vectors (u_1, u_2, u_3), (v_1, v_2, v_3), and (w_1, w_2, w_3)

$$= \left| det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right|$$

Example: Find the area of the parallelogram determined by the vectors (1, 2) and (3, 4).

Example: Find the area of the parallelepiped determined by vectors (1, 4, 5), (2, 10, 0), & (3, 0, 6)

Recall how row operations affect the determinant:

$$\text{If } A \xrightarrow[R_i \to cR_i]{} B, \text{ then } detB = c(detA).$$

$$\text{If } A \xrightarrow[R_i \leftrightarrow R_j]{} B, \text{ then } detB = -(detA).$$

$$\text{If } A \xrightarrow[R_i + cR_j \to R_i]{} B, \text{ then } detB = detA.$$

Note how row operations affect area:

Area of square determined by vectors (1, 0) & (0, 1):

Area of rectangle determined by vectors (a, 0) & (0, b): [redacted]

Area of rectangle determined by vectors (a, 3a) & (0, b): [redacted]

Area of rectangle determined by vectors (0, a) & (b, 0): [redacted]