Note: In ch. 3 all matrices are SQUARE.

3.1 Defn: $detA = \Sigma \pm a_{1j_1}a_{2j_2}...a_{nj_n}$

2 × 2 short-cut: $det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =$

3 × 3 short-cut: $det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$

Note there is no short-cut for $n \times n$ matrices when $n > 3$.

Definition of Determinant using cofactor expansion

Defn: A_{ij} is the matrix obtained from A by deleting the ith row and the jth column.

Defn: Let $A = (a_{ij})$ by an $n \times n$ square matrix. The determinant of A is

1.) If $n = 1$, $detA = a_{11}$.

2.) If $n > 1$, $detA = \Sigma_{k=1}^{n}(-1)^{1+k}a_{1k}detA_{1k}$

$$= a_{11}detA_{11} - a_{12}detA_{12} + ... + (-1)^{1+n}a_{1n}detA_{1n}$$

Note the above definition is an inductive or recursive definition.

Thm: Let $A = (a_{ij})$ by an $n \times n$ square matrix, $n > 1$. Then expanding along row i,

$$detA = \Sigma_{k=1}^{n}(-1)^{i+k}a_{ik}detA_{ik}.$$

Or expanding along column j,

$$detA = \Sigma_{k=1}^{n}(-1)^{k+j}a_{kj}detA_{kj}.$$

Defn: $detA_{ij}$ is the i, j-minor of A.

$(-1)^{i+j}detA_{ij}$ is the i, j-cofactor of A.

3.2: Properties of Determinants

Thm: If $A \xrightarrow[R_i \to cR_i]{} B$, then $detB = c(detA)$.

Warning note: $det(cA) = c^n detA$.

Thm: If $A \xrightarrow[R_i \leftrightarrow R_j]{} B$, then $detB = -(detA)$.

Thm: If $A \xrightarrow[R_i + cR_j \to R_i]{} B$, then $detB = detA$.

Some Shortcuts:

Thm: If A is an $n \times n$ matrix which is either lower triangular or upper triangular, then $det A = a_{11}a_{22}...a_{nn}$, the product of the entries along the main diagonal.

Cor: $det(I_n) = 1$.

Thm: If a square matrix has a row or column containing all zeros, its determinant is zero.

Thm: If some row (column) of a square matrix A is a scalar multiple of another row (column), then $det A = 0$.

Thm: A square matrix is invertible if and only if $det A \neq 0$.

Thm: Let A be a square matrix. Then the linear system $Ax = b$ has a unique solution for every b if and only if $det A \neq 0$.

Thm: $det AB = (det A)(det B)$.

Cor: $det A^{-1} = \frac{1}{det A}$.

$det(A + B) \neq det A + det B$.

Thm: $det A^T = det A$.

Proof of thm $det AB = (det A)(det B)$:

Lemma 1:
Let M be a square matrix, and let E be an elementary matrix of the same order. Then $det(EM) = (det E)(det M)$.

Lemma 2: Let M be a square matrix, and let $E_1, E_2, ..., E_k$ be elementary matrices of the same order as M. Then $det(E_1 E_2 ... E_k M) = (det E_1)(det E_2)...(det E_k)(det M)$.

Lemma 3:
Let $E_1, E_2, ..., E_k$ be elementary matrices of the same order. Then $det(E_1 E_2 ... E_k) = (det E_1)(det E_2)...(det E_k)$.