Find the inverse of $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$.

Long method: $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} =$

$$\begin{bmatrix} 2x_{11}+3x_{21}+4x_{31} & 2x_{12}+3x_{22}+4x_{32} & 2x_{13}+3x_{23}+4x_{33} \\ 4x_{11}+5x_{21}+6x_{31} & 4x_{12}+5x_{22}+6x_{32} & 4x_{13}+5x_{23}+fx_{33} \\ 6x_{11}+7x_{21}+9x_{31} & 6x_{12}+7x_{22}+9x_{32} & 6x_{13}+7x_{23}+9x_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So solve,

$$\begin{bmatrix} 2 & 3 & 4 & 1 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 0 \end{bmatrix} \text{ for } x_{11}, x_{21}, x_{31}.$$

$$\begin{bmatrix} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 1 \\ 6 & 7 & 9 & 0 \end{bmatrix} \text{ for } x_{12}, x_{22}, x_{32}.$$

$$\begin{bmatrix} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 1 \end{bmatrix} \text{ for } x_{13}, x_{23}, x_{33}.$$

Or shorter method, solve

$$\begin{bmatrix} 2 & 3 & 4 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 6 & 7 & 9 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 4 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 6 & 7 & 9 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow (R_2 - 2R_1 \to R_2, R_3 - 3R_1 \to R_3)$$

$$\begin{bmatrix} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -2 & -3 & -3 & 0 & 1 \end{bmatrix}$$

$$\downarrow (-R_2 \to R_2)$$

$$\begin{bmatrix} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & -2 & -3 & -3 & 0 & 1 \end{bmatrix}$$

$$\downarrow (R_3 + 2R_2 \to R_3)$$

$$\begin{bmatrix} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix}$$

$$\downarrow (R_1 - 4R_3 \to R_1, R_2 - 2R_3 \to R_2)$$

$$\begin{bmatrix} 2 & 3 & 0 & -3 & 8 & -4 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix}$$

$$\downarrow (R_1 - 3R_2 \to R_1)$$

$$\begin{bmatrix} 2 & 0 & 0 & -3 & -1 & 2 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix} \xrightarrow[(\frac{1}{2}R_1 \to R_1)]{} \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix}$$

Thus $\begin{bmatrix} 2 & 3 & 4 & 1 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 0 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$,

so $(x_{11}, x_{21}, x_{31}) = (-\frac{3}{2}, 0, 1)$.

$\begin{bmatrix} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 1 \\ 6 & 7 & 9 & 0 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$,

so $(x_{12}, x_{22}, x_{32}) = (\frac{-1}{2}, 3, -2)$.

$\begin{bmatrix} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$,

so $(x_{13}, x_{23}, x_{33}) = (1, -2, 1)$.

Shortest method:

Note that if $[A|I]$ is row equivalent to $[I|B]$, then $B = A^{-1}$.

Thus the inverse of $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$ is $\begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Check answer: $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix} =$

WHEN DOES A^{-1} EXIST?

Solve
$$\begin{aligned} 2x + 3y + 4z &= 0 \\ 4x + 5y + 6z &= 0 \\ 6x + 7y + 9z &= 0 \end{aligned}$$

Solve
$$\begin{aligned} 2x + 3y + 4z &= 0 \\ 4x + 5y + 6z &= 2 \\ 6x + 7y + 9z &= 1 \end{aligned}$$

Elementary matrices and linear systems.

Definition: A matrix is called an elementary matrix if it can be obtained from an identity matrix by exactly one elementary row operation.

Examples:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow kR_1]{} \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 + kR_1 \leftrightarrow R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & j \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} =$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \xrightarrow[R_1 \leftrightarrow kR_1]{} \begin{bmatrix} ka & kb & kc \\ d & e & f \\ g & h & j \end{bmatrix}$$

$$\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} =$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \xrightarrow[R_2 + kR_1 \leftrightarrow R_2]{} \begin{bmatrix} a & b & c \\ d + ka & e + kb & f + kc \\ g & h & j \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} =$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

Note the opposite of exchanging row one with row two is exchanging row two with row one. That is

$$\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \xrightarrow[R_1 \leftrightarrow kR_1]{} \begin{bmatrix}k & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Note the opposite of multiplying row one by k is multiplying row one by $\frac{1}{k}$. That is,

$$\begin{bmatrix}\frac{1}{k} & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}k & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \xrightarrow[R_2 + kR_1 \leftrightarrow R_2]{} \begin{bmatrix}1 & 0 & 0\\k & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Note the opposite of adding k row 1 to row 2 is subtracting k row 1 from row 2. That is,

$$\begin{bmatrix}1 & 0 & 0\\-k & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\k & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Note $A \sim I$ if and only if A is invertible.