Math 2418 Linear Algebra Exam #2 PART A Name: ____________________

October 31, 2001 SHOW ALL WORK Circle one: Wednesday/Thursday

[20] 1.) Find the QR-decomposition of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & 3 & 10 \end{bmatrix}$.

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad <\mathbf{v_1},\ \mathbf{v_1}> = <\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}> = (1)(1) + (0)(0) + (0)(0) + (0)(0) = 1, \quad ||\mathbf{v_1}|| = \sqrt{1} = 1$$

$$<\mathbf{v_1},\ \mathbf{u_2}> = <\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}> = (1)(1) + (0)(1) + (0)(0) + (0)(3) = 1$$

Thus $proj_{\mathbf{v_1}}\mathbf{u_1} = \frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Hence $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}$

$$<\mathbf{v_2},\ \mathbf{v_2}> = <\begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}> = (0)(0) + (1)(1) + (0)(0) + (3)(3) = 10 \qquad ||\mathbf{v_2}|| = \sqrt{< v_2,\ v_2 >} = \sqrt{10}$$

$$<\mathbf{v_1},\ \mathbf{u_3}> = <\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 5 \\ 10 \end{bmatrix}> = (1)(2) + (0)(0) + (0)(5) + (0)(10) = 2$$

$$<\mathbf{v_2},\ \mathbf{u_3}> = <\begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 5 \\ 10 \end{bmatrix}> = (0)(2) + (1)(0) + (0)(5) + (3)(10) = 30$$

Thus $proj_{\mathbf{v_1}}\mathbf{u_1} = \frac{2}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{30}{10}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} == \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 9 \end{bmatrix}$. Hence $\mathbf{v_3} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 10 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 5 \\ 1 \end{bmatrix}$

$$<\mathbf{v_3},\ \mathbf{v_3}> = <\begin{bmatrix} 0 \\ -3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 5 \\ 1 \end{bmatrix}> = (0)(0) + (-3)(-3) + (5)(5) + (1)(1) = 35. \ ||\mathbf{v_3}|| = \sqrt{< v_3,\ v_3 >} = \sqrt{35}$$

Thus $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{35}} \\ 0 & 0 & \frac{5}{\sqrt{35}} \\ 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{35}} \end{bmatrix}$ and $R = Q^T A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \\ 0 & -\frac{3}{\sqrt{35}} & \frac{5}{\sqrt{35}} & \frac{1}{\sqrt{35}} \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & 3 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & \frac{10}{\sqrt{10}} & \frac{30}{\sqrt{10}} \\ 0 & 0 & \frac{35}{\sqrt{35}} \end{bmatrix}$

Answer: $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{35}} \\ 0 & 0 & \frac{5}{\sqrt{35}} \\ 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{35}} \end{bmatrix}$ $R = \begin{bmatrix} 1 & 1 & 2 \\ 0 & \frac{10}{\sqrt{10}} & \frac{30}{\sqrt{10}} \\ 0 & 0 & \frac{35}{\sqrt{35}} \end{bmatrix}$

[2] 1b.) An orthonormal basis for the column space of A is $\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{3}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \\ \frac{1}{\sqrt{35}} \end{bmatrix}\}$.

2.) Let $W = span\{1+t,\ 1-3t\}$. Note that $\{1+t,\ 1-3t\}$ is an orthogonal set.
Using the inner product $< f,\ g >= \int_{-1}^{1} f(t)g(t)dt$, $< 1+t,\ 1+t >= \frac{8}{3}$ and $< 1-3t,\ 1-3t >= 8$.
Using this inner product, find the following:

[2] 2a.) $||1+t|| = \underline{\sqrt{\frac{8}{3}}}$

[2] 2b.) $||1-3t|| = \underline{\sqrt{8}}$

[3] 2c.) $< 7t^5,\ 1+t >= \underline{2}$

$\int_{-1}^{1} 7t^5(1+t) = \int_{-1}^{1} 7t^5 + 7t^6 = \frac{7t^6}{6} + t^7|_{-1}^{1} = 2$

[3] 2d.) $< 7t^5,\ 1-3t >= \underline{-6}$

$\int_{-1}^{1} 7t^5(1-3t) = \int_{-1}^{1} 7t^5 - 21t^6 = \frac{7t^6}{6} - 3t^7|_{-1}^{1} = -6$

[3] 2e.) If $\mathbf{v} = 7t^5$, $proj_W \mathbf{v} = \underline{3t}$

$2 \cdot \frac{3}{8}(1+t) - \frac{6}{8}(1-3t) = \frac{6}{8} + \frac{6}{8}t - \frac{6}{8} + \frac{18}{8}t = \frac{24}{8}t = 3t$

[3] 2f.) Is $7t^5$ in W? $\underline{NO}$

[3] 2g.) A vector in the orthogonal complement of W is $\underline{7t^5 - 3t}$.

[4] 2h.) Find an orthogonal basis for $\{1+t,\ 1-3t,\ 7t^5\}$ which includes $1+t$ and $1-3t$.

$\{1+t,\ 1-3t, 7t^5 - 3t\}$

[2] 2i.) If $\mathbf{u} = t$, $proj_W \mathbf{u} = \underline{t}$

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[18] 3a.) The following matrix has only one eigenvalue: $A = \begin{bmatrix} 2 & 3 & 3 & 3 \\ 0 & 2 & 1 & 8 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Find the eigenvalue and a basis for the eigenspace corresponding to this eigenvalue.

Since this is a diagonal matrix, the eigenvalues are the diagonal entries. Thus $\lambda = 2$ is the only eigenvalue of A.

$$(2I - A) = \begin{bmatrix} 0 & -3 & -3 & -3 \\ 0 & 0 & -1 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s \\ 7t \\ -8t \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 7 \\ -8 \\ 1 \end{bmatrix}$$

Answer 3a) Eigenvalue: $\lambda =$ <u>2</u>

Basis for Eigenspace corresponding to λ: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ -8 \\ 1 \end{bmatrix} \right\}$

[3] 3b.) List 3 eigenvectors of A corresponding to λ_2: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ -8 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -8 \\ 1 \end{bmatrix}$

Note any vector of the form $\begin{bmatrix} s \\ 7t \\ -8t \\ t \end{bmatrix}$ is an eigenvector of A.

[3] 3c.) List two vectors in R^3 which are not eigenvectors of A: $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

For this matrix, If a vector is NOT of the form $\begin{bmatrix} s \\ 7t \\ -8t \\ t \end{bmatrix}$, then it is NOT an eigenvector of A.

The vector $\mathbf{0}$ is never an eigenvector.

[10] 4.) Let $< (u_1, u_2),\ (v_1, v_2) > =\ 4u_1v_1 + 2u_2v_2$.
Show that this operation satisfies $< \mathbf{u} + \mathbf{v},\ \mathbf{w} > = < \mathbf{u},\ \mathbf{w} > + < \mathbf{v},\ \mathbf{w} >$

Let $\mathbf{u} = (u_1,\ u_2)$, $\mathbf{v} = (v_1,\ v_2)$, and $\mathbf{w} = (w_1,\ w_2)$. Then

$< \mathbf{u} + \mathbf{v},\ \mathbf{w} > = < (u_1,\ u_2) + (v_1,\ v_2),\ (w_1,\ w_2) > = < (u_1 + v_1,\ u_2 + v_2),\ (w_1,\ w_2) >$
$=\ 4(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2\ =\ 4u_1w_1 + 4v_1w_1 + 2u_2w_2 + 2v_2w_2$

$< \mathbf{u},\ \mathbf{w} > + < \mathbf{v},\ \mathbf{w} > = < (u_1,\ u_2),\ (w_1,\ w_2) > + < (v_1,\ v_2),\ (w_1,\ w_2) >$
$=\ 4u_1w_1 + 2u_2w_2 + 4v_1w_1 + 2v_2w_2\ =\ 4u_1w_1 + 4v_1w_1 + 2u_2w_2 + 2v_2w_2$

Thus, $< \mathbf{u} + \mathbf{v},\ \mathbf{w} > = < \mathbf{u},\ \mathbf{w} > + < \mathbf{v},\ \mathbf{w} >$

5.) Circle T for True or F for False.

[3] 5a.) If $\{\mathbf{v_1}, ..., \mathbf{v_n}\}$ is an orthogonal set of vectors, then $\{\mathbf{v_1}, ..., \mathbf{v_n}\}$ is linearly independent. T

[3] 5b.) If λ is not an eigenvalue of A, then the linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has only the trivial solution T

[3] 5c.) If the characteristic equation of A is $p(\lambda) = \lambda(\lambda - 5)(\lambda - 8)^2$, then A is invertible. F

6.) Circle the correct answer

[3] 6a.) If W is a line in R^2, then $W^{\perp}$ is
iii.) a line.

[3] 6b.) If W is a line in R^3, then $W^{\perp}$ is
iv.) a 2-dimensional plane.

[3] 7a.) If $\mathbf{u}$ is in W, then $proj_W\mathbf{u} = \underline{\ \mathbf{u}\ }$

[3] 7b.) If $\mathbf{u}$ is in $W^{\perp}$, then $proj_W\mathbf{u} = \underline{\ \mathbf{0}\ }$