Math 2418 Linear Algebra Exam #1
September 25, 2002 SHOW ALL WORK

1.) Solve the following system of equations.

$$\begin{aligned} -x_2 + x_3 + 2x_4 &= b_1 \\ 2x_1 + 6x_3 &= b_2 \\ 4x_1 + 3x_2 + 8x_3 + x_5 &= b_3 \\ x_3 - 6x_4 - x_5 &= b_4 \end{aligned}$$

[10] 1a.) when $(b_1, b_2, b_3, b_4) = (1, 0, 1, -4)$.

Answer 1a.) $\underline{(12 - 18s - 3t, -5 + 8s + t, -4 + 6s + t, s, t)}$

[3] 1b.) when $(b_1, b_2, b_3, b_4) = (2, 0, 2, -8)$.

Answer 1b.) $\underline{(24 - 18s - 3t, -10 + 8s + t, -8 + 6s + t, s, t)}$

[3] 1c.) Find two vectors (b_1, b_2, b_3, b_4) for which there does not exist a solution to the above system of equations.

Any such that $b_4 + b_3 - 2b_2 + 3b_1 \neq 0$. For example

Answer 1c.) $\underline{(1, 0, 0, 0), (2, 0, 0, 0)}$

[3] 1d.) If A is the coefficient matrix for the above system of equations, find two vectors, $\mathbf{x}$, such that $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}$

Choose any values for s, t in $(12 - 18s - 3t, -5 + 8s + t, -4 + 6s + t, s, t)$. For example

Answer 1c.) $\underline{(12, -5, -4, 0, 0), (-6, 3, 2, 1, 0)}$

(see next page for room for scratch work)

Scratch work page for solving

$$-x_2 + x_3 + 2x_4 = b_1$$
$$2x_1 + 6x_3 = b_2$$
$$4x_1 + 3x_2 + 8x_3 + x_5 = b_3$$
$$x_3 - 6x_4 - x_5 = b_4$$

when $(b_1, b_2, b_3, b_4) = (1, 0, 1, -4)$. or $(b_1, b_2, b_3, b_4) = (2, 0, 2, -8)$.

$$\begin{bmatrix} 0 & -1 & 1 & 2 & 0 & 1 & 2 & b_1 \\ 2 & 0 & 6 & 0 & 0 & 0 & 0 & b_2 \\ 4 & 3 & 8 & 0 & 1 & 1 & 2 & b_3 \\ 0 & 0 & 1 & -6 & -1 & -4 & -8 & b_4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 6 & 0 & 0 & 0 & 0 & b_2 \\ 0 & -1 & 1 & 2 & 0 & 1 & 2 & b_1 \\ 4 & 3 & 8 & 0 & 1 & 1 & 2 & b_3 \\ 0 & 0 & 1 & -6 & -1 & -4 & -8 & b_4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 6 & 0 & 0 & 0 & 0 & b_2 \\ 0 & -1 & 1 & 2 & 0 & 1 & 2 & b_1 \\ 0 & 3 & -4 & 0 & 1 & 1 & 2 & b_3 - 2b_2 \\ 0 & 0 & 1 & -6 & -1 & -4 & -8 & b_4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 6 & 0 & 0 & 0 & 0 & b_2 \\ 0 & -1 & 1 & 2 & 0 & 1 & 2 & b_1 \\ 0 & 0 & -1 & 6 & 1 & 4 & 8 & b_3 - 2b_2 + 3b_1 \\ 0 & 0 & 1 & -6 & -1 & -4 & -8 & b_4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 6 & 0 & 0 & 0 & 0 & b_2 \\ 0 & -1 & 1 & 2 & 0 & 1 & 2 & b_1 \\ 0 & 0 & -1 & 6 & 1 & 4 & 8 & b_3 - 2b_2 + 3b_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_4 + b_3 - 2b_2 + 3b_1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 36 & 6 & 24 & 48 \\ 0 & -1 & 0 & 8 & 1 & 5 & 10 \\ 0 & 0 & -1 & 6 & 1 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 18 & 3 & 12 & 24 \\ 0 & 1 & 0 & -8 & -1 & -5 & -10 \\ 0 & 0 & 1 & -6 & -1 & -4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(x_1, x_2, x_3, x_4, x_5) = (12 - 18s - 3t, -5 + 8s + t, -4 + 6s + t, s, t)$$

$$(x_1, x_2, x_3, x_4, x_5) = (24 - 18s - 3t, -10 + 8s + t, -8 + 6s + t, s, t)$$

[12] 2.) Given that $\begin{bmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$,

solve the following system of equations by LU factorization

$$\begin{gathered} 3x_1 - 6x_2 - 3x_3 = -3 \\ 2x_1 + 6x_3 = 22 \\ -4x_1 + 7x_2 + 4x_3 = 2 \end{gathered}$$

LUx = b

Let Ux = y. Then Ly = b.

1.) solve Ly = b

$$\begin{bmatrix} 3 & 0 & 0 & -3 \\ 2 & 4 & 0 & 22 \\ -4 & -1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 2 & 0 & 11 \\ -4 & -1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 12 \\ 0 & -1 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 6 \\ 0 & -1 & 2 & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Hence $(y_1, y_2, y_3) = (-1, 6, 2)$.

2.) solve Ux = y

$$\begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

Answer: $\underline{(5, 2, 2)}$

[12] 3.) Find an LU factorization of $A = \begin{bmatrix} 3 & 3 & 0 & 9 \\ 1 & 2 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & 6 & 1 \end{bmatrix}$

U: $\begin{bmatrix} 3 & 3 & 0 & 9 \\ 1 & 2 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 3 \\ 1 & 2 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 6 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 6 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 6 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

L: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix}$

Check $LU = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 6 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 & 9 \\ 1 & 2 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & 6 & 1 \end{bmatrix}$

$$L = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 6 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

4.) Find the determinant of $A = \begin{bmatrix} 3 & 3 & 0 & 3 \\ 1 & 2 & 6 & 0 \\ 0 & 5 & 2 & 0 \\ 2 & 0 & 2 & 3 \end{bmatrix}$

$$\begin{vmatrix} 3 & 3 & 0 & 3 \\ 1 & 2 & 6 & 0 \\ 0 & 5 & 2 & 0 \\ 2 & 0 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 3 & 0 & 3 \\ 1 & 2 & 6 & 0 \\ 0 & 5 & 2 & 0 \\ -1 & -3 & 2 & 0 \end{vmatrix} = \text{-3} \begin{vmatrix} 1 & 2 & 6 \\ 0 & 5 & 2 \\ -1 & -3 & 2 \end{vmatrix} = \text{-3} \begin{vmatrix} 1 & 2 & 6 \\ 0 & 5 & 2 \\ 0 & -1 & 8 \end{vmatrix}$$

$$= \text{-3} \begin{vmatrix} 5 & 2 \\ -1 & 8 \end{vmatrix} = \text{-3(40 - (-2))} = \text{-3(42)} = \text{-126}$$

[10] 4a.) $det(A) = \underline{-126}$

[2] 4b.) Is A invertible? $\underline{yes}$

[4] 4c.) Solve $A\mathbf{x} = \mathbf{0}$. Answer 4c.): $\underline{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$

Since A is invertible, $A\mathbf{x} = \mathbf{0}$ has a unique solution. Since the zero vector is always a solution to $A\mathbf{x} = \mathbf{0}$ and A is invertible, the zero vector is the unique solution ($A\mathbf{x} = \mathbf{0}$, A invertible implies $A^{-1}A\mathbf{x} = A^{-1}\mathbf{0}$ which implies $\mathbf{x} = \mathbf{0}$).

NOTE: In the future when the answer is the zero vector, you are required to specify which zero vector. For example, the answer to 4c should be a 4×1 matrix (the 4×1 zero matrix). Stating the $\mathbf{x} = \mathbf{0}$ will result in only partial credit. For full credit, you must state $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$)

[10] 5.) Prove by giving a specific example that $AB = AC$ does not imply $B = C$.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

6.) Let $D = \begin{bmatrix} 2 & 1 & 60 & 4 & 7 & 6 & 3 \\ 0 & 5 & 89 & 8 & 9 & 5 & 5 \\ 0 & 0 & 10 & 9 & 2 & 4 & 7 \\ 0 & 0 & 0 & 1 & 4 & 7 & 8 \\ 0 & 0 & 0 & 0 & 3 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

$det\ D = 2(5)(10)(1)(3)(1)(1) = 300$

[3] 6a.) $det\ D = \underline{300}$.

[3] 6b.) How many solutions does $D\mathbf{x} = \mathbf{0}$ have? $\underline{one}$.

[3] 6c.) Is D invertible? $\underline{yes}$.

[3] 6d.) $det\ E = \underline{0}$.

[3] 6e.) How many solutions does $E\mathbf{x} = \mathbf{0}$ have? $\underline{infinite}$.

[3] 6f.) Is E invertible? $\underline{No}$.

[5] 7.) If $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 23 & 13 & 20 & 44 & 73 & 65 & 34 & 83 & 70 & 46 & 53 & 49 \\ 93 & 54 & 82 & 83 & 94 & 54 & 85 & 93 & 28 & 94 & 72 & 45 \\ 59 & 74 & 10 & 92 & 20 & 49 & 67 & 46 & 26 & 34 & 79 & 35 \end{bmatrix}$,

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 82 \\ 10 \end{bmatrix} = \begin{bmatrix} 90 \\ 10 \end{bmatrix}$$

then the third column of $AB = \underline{\begin{bmatrix} 90 \\ 10 \end{bmatrix}}$.

[3] 8.) If A is a 2×2 matrix and $det\ A = 10$, then $det\ [3A] = \underline{90}$.

[3] 9.) If A is a 4×4 matrix and $det\ A = 10$, then $det\ [3A] = \underline{810}$.

[3] 10.) If A is a 4×4 matrix, $det\ A = 10$ and B is a matrix obtained from A by multiplying the 2nd row of A by 3, then $det\ B = \underline{30}$.

[5] 11.) A unit vector in the same direction as the vector (2, 10, 5) is $\underline{(\frac{2}{\sqrt{129}}, \frac{10}{\sqrt{129}}, \frac{5}{\sqrt{129}})}$.

$\sqrt{4 + 100 + 25} = \sqrt{129}$

12.) Circle T for true and F for False.

[4] 12a.) The LU method for solving equations can only be applied to square matrices.
F

[4] 12b.) A system of equations with more variables than equations cannot have a unique solution.
T

[4] 12c.) A system of equations with more variables than equations can have no solution.
T

[4] 12d.) If $A\mathbf{x} = \mathbf{b}$ has a unique solution, then A$\mathbf{x} = \mathbf{0}$ has a unique solution. T

[4] 12e.) If $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions, then $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions.
F