Math 2418 Linear Algebra Exam #1 Name: ______________________

Oct 1, 2001 SHOW ALL WORK Circle one: Wednesday/Thursday

[14] 1.) $det \begin{bmatrix} 4 & 8 & 3 & 4 \\ 5 & 0 & 10 & 4 \\ 2 & 4 & 3 & 1 \\ 3 & 0 & 2 & 1 \end{bmatrix} = \underline{244}$

$$det \begin{bmatrix} 4 & 8 & 3 & 4 \\ 5 & 0 & 10 & 4 \\ 2 & 4 & 3 & 1 \\ 3 & 0 & 2 & 1 \end{bmatrix} = det \begin{bmatrix} 0 & 0 & -3 & 2 \\ 5 & 0 & 10 & 4 \\ 2 & 4 & 3 & 1 \\ 3 & 0 & 2 & 1 \end{bmatrix} = -4det \begin{bmatrix} 0 & -3 & 2 \\ 5 & 10 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= -4[-(-3)det \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} + 2det \begin{bmatrix} 5 & 10 \\ 3 & 2 \end{bmatrix}]$$

$$= -4[-(-3)(5-12) + 2(10-30)] = -4[-(-3)[-7] + 2(-20)] = -4[-21-40] = -4[-61] = 244$$

[6] 2a.) The orthogonal projection of the vector (3, 5) onto the vector (1, 2) is $\underline{(\frac{13}{5}, \frac{26}{5})}$

$(3,5) \cdot (1,2) = 3 + 10 = 13$
$(1,2) \cdot (1,2) = 1 + 4 = 5$
$proj_a u = \frac{13}{5}(1,2) = (\frac{13}{5}, \frac{26}{5})$

[6] 2b.) The orthogonal component of the vector (3, 5) orthogonal to (1, 2) is $\underline{(\frac{2}{5}, \frac{-1}{5})}$

$(3,5) - (\frac{13}{5}, \frac{26}{5}) = (\frac{2}{5}, \frac{-1}{5})$

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[12] 3.) Solve each of the following system of linear equations by using Gauss-Jordan elimination.

3a.)
$$\begin{aligned} x_2 + 4x_3 &= 1 \\ 3x_1 + 2x_2 &= 0 \\ 6x_1 + 5x_2 + 4x_3 &= 1 \end{aligned}$$

3b.)
$$\begin{aligned} x_2 + 4x_3 &= 1 \\ 3x_1 + 2x_2 &= 0 \\ 6x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 & 1 & 4 & 1 & 1 \\ 3 & 2 & 0 & 0 & 0 \\ 6 & 5 & 4 & 1 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 \\ 6 & 5 & 4 & 1 & 0 \end{bmatrix} \xrightarrow[R_3 - 2R_1 \to R_3]{} \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 \\ 0 & 1 & 4 & 1 & 0 \end{bmatrix} \xrightarrow[R_3 - R_2 \to R_3]{}$$

$$\begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \xrightarrow[R_1 - 2R_2 \to R_1]{} \begin{bmatrix} 3 & 0 & -8 & -2 & -2 \\ 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \xrightarrow[\frac{R_1}{3} \to R_1]{} \begin{bmatrix} 1 & 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Answer 3a.) $x_1 = -\frac{2}{3} + \frac{8}{3}t,\ x_2 = 1 - 4t,\ x_3 = t$ 3b.) *no solution*

[2] 3c.) If A = coefficient matrix in 1a, does A^{-1} exist? *NO*

[2] 3d.) If A = coefficient matrix in 1a, *det* $A =$ 0

[1] 3d.) The answer to 1a is a hyperplane that lives in R^m where $m =$ 3.

[1] 3e.) The dimension of the hyperplane in 1a is 1.

[5] 3f.) An equation of the hyperplane in 1a in point-parallel vector form is

$$\mathbf{x} = (-\frac{2}{3}, 1, 0) + t(\frac{8}{3}, -4, 1)$$

[3] 3g.) Using different numbers, an equivalent equation of the hyperplane in 1a in point-parallel vector form is

$$\mathbf{x} = (10, -15, 4) + t(8, -12, 3)$$

Note many other answers are possible. I multiplied $(\frac{8}{3}, -4, 1)$ by 3 to get the vector (8, -12, 3) which also describes the direction of this line. Any scalar multiple (except 0) of $(\frac{8}{3}, -4, 1)$ would also describe the direction of this line. I set $t = 4$ to find that (10, -15, 4) is a point on this line. Any other point on this line could also be used (can choose any value for t to find another point on this line).

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[20] 4.) Find and use an LU factorization to solve:

$$\begin{bmatrix} 4 & 8 \\ 3 & 7 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 \\ 3 & 7 \end{bmatrix} \xrightarrow{\frac{R_1}{4} \to R_1} \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \xrightarrow{R_2 - 3R_1 \to R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Thus $U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $L = \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}$. Check $LU = \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 3 & 7 \end{bmatrix}$

Solve $L(Ux) = b$. Let $Ux = y$. Then 1) Solve $Ly = b$ for y and 2) Solve $Ux = y$ for x.

Step 1) Solve $Ly = b$ for y

$$\begin{bmatrix} 4 & 0 & 4 \\ 3 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{R_1}{4} \to R_1} \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 - 3R_1 \to R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

Thus $y = (1, -3)$

Step 2) Solve $Ux = y$ for x.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_1 - 2R_1 \to R_1} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \end{bmatrix}$$

Thus $x = (7, -3)$

Check $\begin{bmatrix} 4 & 8 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Answer: $L = \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\mathbf{x} = (7, -3)$

5.) Circle T for True or F for False.

[3] a.) Suppose a homogeneous system of 3 linear equations with 2 unknowns has exactly one solution, then any system with the same coefficients will also have exactly one solution. F

[3] b.) Suppose a homogeneous system of 3 linear equations with 3 unknowns has exactly one solution, then any system with the same coefficients will also have exactly one solution. T

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[10] 6.) Suppose $\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & -15 & -2 \\ 0 & 1 & 0 \\ -1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solve the following system of equations using the method of inverses:

$$\begin{aligned} 3x_1 - 15x_2 - 2x_3 &= 10 \\ x_2 &= 0 \\ -x_1 + 5x_2 + x_3 &= 2 \end{aligned}$$

Solve $\begin{bmatrix} 3 & -15 & -2 \\ 0 & 1 & 0 \\ -1 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 2 \end{bmatrix}$

$Ax = b$ implies $A^{-1}Ax = A^{-1}b$. Thus $x = A^{-1}b$.

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 16 \end{bmatrix}$$

Answer 6.) $\underline{(14, 0, 16)}$

[5] 7a.) Given the line $x_1 = 3 + 5t, x_2 = 1 + t, x_3 = 4 + 2t$, then $\mathbf{x} = (3, 1, 4) + t(5, 1, 2)$ and

a point on the line is $\underline{(3, 1, 4)}$

and a vector describing the direction of the line is $\underline{(5, 1, 2)}$.

[5] 7b.) A vector perpendicular to (1, 4, 0) and (5, 2, 1) is $\underline{(4, -1, -18)}$.

$$det \begin{bmatrix} i & j & k \\ 1 & 4 & 0 \\ 5 & 2 & 1 \end{bmatrix} = i(4-0) - j(1-0) + k(2-20) = 4i - j - 18k = (4, -1, -18)$$

[5] 7c.) Find an equation for the plane in point-parallel form that contains the line $x_1 = 3 + 5t, x_2 = 1 + t, x_3 = 4 + 2t$ and is parallel to the line of intersection of the planes $x_1 + 4x_2 + 1 = 0$ and $5x_1 + 2x_3 + x_3 = 0$ (Hint: use the point in 7a and the vectors in 7a and 7b.

Answer 7c.) $\underline{\mathbf{x} = (3, 1, 4) + t(5, 1, 2) + s(4, -1, -18)}$.

Alternate method:

$$\begin{bmatrix} 1 & 4 & 0 & -1 \\ 5 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 & -1 \\ 0 & -18 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 & -1 \\ 0 & 1 & -\frac{1}{18} & -\frac{5}{18} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{4}{18} & \frac{2}{18} \\ 0 & 1 & -\frac{1}{18} & -\frac{5}{18} \end{bmatrix}$$

Thus in parametric form, the line of intersection of the planes $x_1+4x_2+1 = 0$ and $5x_1+2x_3+x_3 = 0$ is $x_1 = \frac{2}{18} - \frac{4}{18}t,\ x_2 = -\frac{5}{18} + \frac{1}{18}t,\ x_3 = t$.

In point parallel vector form: $\mathbf{x} = (\frac{2}{18}, -\frac{5}{18}, 0) + t(-\frac{4}{18}, \frac{1}{18}, 1)$

Thus $(-\frac{4}{18}, \frac{1}{18}, 1)$ is a vector describing the direction of this line.