Math 2418 Linear Algebra Exam #3 Part A
October 20, 2002 SHOW ALL WORK

[7] 1.) Show that the inner product $<(u_1, u_2), (v_1, v_2)> = 3u_1v_1 + 4u_2v_2$ satisfies the additivity axiom of an inner product: $<\mathbf{u}+\mathbf{v}, \mathbf{w}> = <\mathbf{u}, \mathbf{w}> + <\mathbf{v}, \mathbf{w}>$

$<\mathbf{u}+\mathbf{v}, \mathbf{w}> = <(u_1, u_2) + (v_1, v_2), (w_1, w_2)> = <(u_1+v_1, u_2+v_2), (w_1, w_2)> = 3(u_1 + v_1)w_1 + 4(u_2+v_2)w_2 = 3u_1w_1 + 3v_1w_1 + 4u_2w_2 + 4v_2w_2$

$<\mathbf{u}, \mathbf{w}> + <\mathbf{v}, \mathbf{w}> = 3u_1w_1 + 4u_2w_2 + 3v_1w_1 + 4v_2w_2 = 3u_1w_1 + 3v_1w_1 + 4u_2w_2 + 4v_2w_2$

Hence, $<\mathbf{u}+\mathbf{v}, \mathbf{w}> = <\mathbf{u}, \mathbf{w}> + <\mathbf{v}, \mathbf{w}>$

[7] 2.) Show that $<(u_1, u_2), (v_1, v_2)> = 3u_2v_1 + 4u_1v_2$ is not an inner product by giving a specific example (using only real numbers) and showing that the inner product definition or a consequence of it does not hold.

$<(1,0),(0,1)> = 3(0)(0) + 4(1)(1) = 4$

$<(0,1),(1,0)> = 3(1)(1) + 4(0)(0) = 3$

Thus, $<(1,0),(0,1)> \neq <(0,1),(1,0)>$

Alternate answer: $<(1,0),(1,0)> = 3(0)(1) + 4(1)(0) = 0$ but $(1,0) \neq (0,0)$.

Note there are many alternate answers.

Math 2418 Linear Algebra Exam #3 Part B
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[19] 3.) The matrix $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix}$ has two eigenvalues. Find these eigenvalues and bases for the corresponding eigenspaces (Note: this problem continues on the next page).

$$\lambda I - A = \begin{vmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda - 1 & -4 \\ 0 & 0 & -2 & \lambda - 8 \end{vmatrix} = \lambda^2[(\lambda - 1)(\lambda - 8) - 8] = \lambda^2(\lambda^2 - 9\lambda) = \lambda^3(\lambda - 9)$$

Thus $\lambda = 0$ is an eigenvalue of A with algebraic multiplicity 3 and $\lambda = 9$ is an eigenvalue of A with algebraic multiplicity 1.

$\lambda = 0$: Solve $(\lambda I - A)\mathbf{x} = \mathbf{0}$

$\lambda I - A = 0(I) - A = A$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -4 & 0 \\ 0 & 0 & -2 & -8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r \\ s \\ -4t \\ t \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

An eigenvalue of A is <u>0</u>

A basis corresponding to this eigenvalue is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

List 5 eigenvectors corresponding to this eigenvalue.

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -4 \\ 1 \end{bmatrix}$ or any linear combination of the basis vectors except the zero vector (recall that the zero vector is by definition never an eigenvector even though it is always in the eigenspace - since the zero vector is in every vector space).

3 cont.)

$\lambda = 0$: Solve $(\lambda I - A)\mathbf{x} = \mathbf{0}$

$\lambda I - A = 9(I) - A$

$$\begin{bmatrix} 9 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 8 & -4 & 0 \\ 0 & 0 & -2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$$

Thus, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$

A second eigenvalue of A is $\underline{\ 9\ }$

A basis corresponding to this second eigenvalue is (circle your answer)

$\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$ or any multiple of this vector such as $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

4.) If $C \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $C \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $C \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 \\ -6 \end{bmatrix}$,

[3] 4a.) The eigenvalue(s) of C is/are $\underline{\ 0,\ -2\ }$

[3] 4b.) Three eigenvectors of C are $\underline{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}}$

Note any multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of C (corresponding to eigenvalue 0) and any multiple of $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ is an eigenvector of C (corresponding to eigenvalue -2)

[2] 4c.) A vector which is not an eigenvector of C is $\underline{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}$

Any vector which is not a multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ nor a multiple of $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ is not an eigenvector of C

5.) Circle T for true and F for false.

[4] 5a.) Matlab always finds an exactly correct answer when solving $A\mathbf{x} = \mathbf{b}$ when the columns of A are linearly independent. F

[4] 5b.) If $\lambda = 0$ is an eigenvalue of A, then A is NOT invertible. T

[15] 6a.) Find a QR factorization of $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \\ 1 & 4 \end{bmatrix}$.

Gram-Schmidt:

$(2, 2, 1) \cdot (3, 1, 4) = 6 + 2 + 4 = 12$

$(2, 2, 1) \cdot (2, 2, 1) = 4 + 4 + 1 = 9$

$$proj_{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \frac{12}{9} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \frac{4}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ \frac{8}{3} \\ \frac{4}{3} \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} \frac{8}{3} \\ \frac{8}{3} \\ \frac{4}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ \frac{8}{3} \end{bmatrix}$$

Check: $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ \frac{8}{3} \end{bmatrix} = \frac{2}{3} - \frac{10}{3} + \frac{8}{3} = 0$

Normalize:

$$\frac{1}{\sqrt{9}} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ \frac{8}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ \frac{8}{3} \end{bmatrix} = \frac{1}{9} + \frac{25}{9} + \frac{64}{9} = \frac{90}{9} = 10$$

$$\frac{1}{\sqrt{10}} \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ \frac{8}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3\sqrt{10}} \\ -\frac{5}{3\sqrt{10}} \\ \frac{8}{3\sqrt{10}} \end{bmatrix}$$

$QR = A$. Hence $R = Q^T QR = Q^T A = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3\sqrt{10}} & -\frac{5}{3\sqrt{10}} & \frac{8}{3\sqrt{10}} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 3\sqrt{10} \end{bmatrix}$

Answer 6a.) $Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{3\sqrt{10}} \\ \frac{2}{3} & -\frac{5}{3\sqrt{10}} \\ \frac{1}{3} & \frac{8}{3\sqrt{10}} \end{bmatrix}$ $\qquad R = \begin{bmatrix} 3 & 4 \\ 0 & 3\sqrt{10} \end{bmatrix}$

[4] 6b.) An orthonormal basis (according to the usual dot product) for the column space of A is

$$\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3\sqrt{10}} \\ -\frac{5}{3\sqrt{10}} \\ \frac{8}{3\sqrt{10}} \end{bmatrix}$$

[2] 6c.) A basis for the nullspace of A^T is

Solve augmented matrix $\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 1 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & \frac{1}{2} & 0 \\ 3 & 1 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & \frac{1}{2} & 0 \\ 0 & -2 & \frac{5}{2} & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{5}{4} & 0 \end{bmatrix} \rightarrow$
$\begin{bmatrix} 1 & 0 & \frac{7}{4} & 0 \\ 0 & 1 & -\frac{5}{4} & 0 \end{bmatrix}$

Hence, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{4}t \\ \frac{5}{4}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{7}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix}$

and thus a basis for the nullspace of A^T is $\{ \begin{bmatrix} -\frac{7}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix} \}$ or $\{ \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix} \}$.

Check:

$(-7, 5, 4) \cdot (2, 2, 1) = -14 + 10 + 4 = 0$

$(-7, 5, 4) \cdot (3, 1, 4) = -21 + 5 + 16 = 0$

QUICKER METHOD:

or note nullspace of A^T is orthogonal to the column space of A. Since these spaces live in R^3 and the column space of A is a 2-dimensional plane, the nullspace of A^T = orthogonal complement of column space of A is a line. A vector which is perpendicular to the column space is $\begin{vmatrix} i & j & k \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = (8 - 1, -(8 - 3), 2 - 6) = (7, -5, -4)$

Thus a basis for the nullspace of A^T is $\{ \begin{bmatrix} 7 \\ -5 \\ -4 \end{bmatrix} \}$ (or any multiple of this vector)

Check:

$(7, -5, -4) \cdot (2, 2, 1) = 14 - 10 - 4 = 0$

$(7, -5, -4) \cdot (3, 1, 4) = 21 - 5 - 16 = 0$

[25] 7.) $\{1+2x+x^2,\ \ 1+2x-5x^2\}$ is an orthogonal set (but not an orthonormal set) given the inner product $< f, g >= \int_{-1}^{1} fg dx$. Use this inner product to solve the following problems.

7a.) The projection of $3x$ onto span$\{1+2x+x^2,\ \ 1+2x-5x^2\}$ = $\underline{\frac{4}{A^2}(1+2x+x^2)+\frac{4}{B^2}(1+2x-5x^2)}$

$\int_{-1}^{1}(3x)(1+2x+x^2)dx = \int_{-1}^{1}(3x+6x^2+3x^3)dx = (\frac{3}{2}x^2+2x^3+\frac{3}{4}x^4)|_{-1}^{1} = 2-(-2) = 4.$

$\int_{-1}^{1}(3x)(1+2x-5x^2) = \int_{-1}^{1}(3x+6x^2-15x^3)dx = (\frac{3}{2}x^2+2x^3-\frac{15}{4}x^4)|_{-1}^{1} = 2-(-2) = 4.$

Thus, if $||1+2x+x^2|| = A$ and $||1+2x-5x^2|| = B$, then the projection of $3x$ onto span$\{1+2x+x^2,\ \ 1+2x-5x^2\} = \frac{4}{A^2}(1+2x+x^2)+\frac{4}{B^2}(1+2x-5x^2)$

7b.) The projection of $15x^2$ onto span$\{1+2x+x^2,\ \ 1+2x-5x^2\}$ = $\underline{\frac{16}{A^2}(1+2x+x^2)-\frac{20}{B^2}(1+2x-5x^2)}$

$\int_{-1}^{1}(15x^2)(1+2x+x^2)dx = \int_{-1}^{1}(15x^2+30x^3+15x^4)dx = (5x^3+\frac{30}{4}x^4+3x^5)|_{-1}^{1} = 5-(-5)+3-(-3) = 16.$

$\int_{-1}^{1}(15x^2)(1+2x-5x^2)dx = \int_{-1}^{1}(15x^2+30x^3-75x^4)dx = (5x^3+\frac{30}{4}x^4-15x^5)|_{-1}^{1} = 5-(-5)-15-15 = -20.$

Thus, if $||1+2x+x^2|| = A$ and $||1+2x-5x^2|| = B$, then the projection of $3x$ onto span$\{1+2x+x^2,\ \ 1+2x-5x^2\} = \frac{16}{A^2}(1+2x+x^2)+\frac{-20}{B^2}(1+2x-5x^2)$

7c.) If possible, write the $3x$ as a linear combination of $\{1+2x+x^2,\ \ 1+2x-5x^2\}$. If not possible, state not possible.

Not possible

7d.) If possible, write the $15x^2$ as a linear combination of $\{1+2x+x^2,\ \ 1+2x-5x^2\}$. If not possible, state not possible.

$\frac{16}{A^2}(1+2x+x^2)+\frac{20}{B^2}(1+2x-5x^2)$

7e.) If $W = span\{1+2x+x^2,\ \ 1+2x-5x^2\}$, find a basis for $W^{\perp}$.

$$3x - [\frac{4}{A^2}(1+2x+x^2)+\frac{4}{B^2}(1+2x-5x^2)]$$

Since $3x$ is not in W, $3x - proj_W(3x)$ is perpendicular to W and thus is in $W^{\perp}$. Since the orthogonal compliment of a 2-dimensional subspace of a 3-dimensional vector space is a 1-dimensional subspace (3 - 2 = 1), $W^{\perp}$ is 1-dimensional.

7f.) Find an orthogonal basis (according to the given inner product) for P_2 which contains the polynomials $1+2x+x^2$ and $1+2x-5x^2$

$$\{1+2x+x^2, 1+2x-5x^2, 3x - [\frac{4}{A^2}(1+2x+x^2)+\frac{4}{B^2}(1+2x-5x^2)]\}$$

This is what you would get by applying the G. S. algorithm, but note you have already done all the steps of G. S. in the preceding problems.

[32] 7.) $\{1+2x+x^2,\ \ 1+2x-5x^2\}$ is an orthogonal set (but not an orthonormal set) given the inner product $< f,g >= \int_{-1}^{1} fg dx$. Use this inner product to solve the following problems.

$\int_{-1}^{1}(1+2x+x^2)(1+2x+x^2)dx = \int_{-1}^{1}(1+4x+6x^2+4x^3+x^4)dx = x+2x^2+2x^3+x^4+\frac{1}{5}x^5|_{-1}^{1} = 1-(-1)+2-(-2)+\frac{1}{5}-(-\frac{1}{5}) = \frac{32}{5}$

Thus, $A^2 = \frac{32}{5}$

$\int_{-1}^{1}(1+2x-5x^2)(1+2x-5x^2)dx = \int_{-1}^{1}(1+4x-6x^2-20x^3+25x^4)dx = x+2x^2-2x^3-5x^4+5x^5|_{-1}^{1} = 1-(-1)-[2-(-2)]+5-(-5) = 2-4+10 = 8$

Thus, $B^2 = 8$

7a.) The projection of $3x$ onto span$\{1+2x+x^2,\ \ 1+2x-5x^2\} = \underline{\frac{9}{8}+\frac{9}{4}x-\frac{15}{8}x^2}$

$\int_{-1}^{1}(3x)(1+2x+x^2)dx = \int_{-1}^{1}(3x+6x^2+3x^3)dx = (\frac{3}{2}x^2+2x^3+\frac{3}{4}x^4)|_{-1}^{1} = 2-(-2) = 4.$

$\int_{-1}^{1}(3x)(1+2x-5x^2) = \int_{-1}^{1}(3x+6x^2-15x^3)dx = (\frac{3}{2}x^2+2x^3-\frac{15}{4}x^4)|_{-1}^{1} = 2-(-2) = 4.$

Thus, the projection of $3x$ onto span$\{1+2x+x^2,\ \ 1+2x-5x^2\} =$

$\frac{4}{\frac{32}{5}}(1+2x+x^2)+\frac{4}{8}(1+2x-5x^2) = \frac{5}{8}(1+2x+x^2)+\frac{4}{8}(1+2x-5x^2) = \frac{9}{8}+\frac{18}{8}x-\frac{15}{8}x^2 = \frac{9}{8}+\frac{9}{4}x-\frac{15}{8}x^2$

7b.) The projection of $15x^2$ onto span$\{1+2x+x^2,\ \ 1+2x-5x^2\} = \underline{15x^2}$

$\int_{-1}^{1}(15x^2)(1+2x+x^2)dx = \int_{-1}^{1}(15x^2+30x^3+15x^4)dx = (5x^3+\frac{30}{4}x^4+3x^5)|_{-1}^{1} = 5-(-5)+3-(-3) = 16.$

$\int_{-1}^{1}(15x^2)(1+2x-5x^2)dx = \int_{-1}^{1}(15x^2+30x^3-75x^4)dx = (5x^3+\frac{30}{4}x^4-15x^5)|_{-1}^{1} = 5-(-5)-15-(15) = -20.$

Thus, the projection of $15x^2$ onto span$\{1+2x+x^2,\ \ 1+2x-5x^2\} = \frac{16}{\frac{32}{5}}(1+2x+x^2)+\frac{-20}{8}(1+2x-5x^2)$

$= \frac{5}{2}(1+2x+x^2)-\frac{5}{2}(1+2x-5x^2) = 15x^2$

7c.) If possible, write the $3x$ as a linear combination of $\{1+2x+x^2,\ \ 1+2x-5x^2\}$. If not possible, state not possible.

Not possible since $3x \neq \frac{9}{8}+\frac{9}{4}x-\frac{15}{8}x^2$

7d.) If possible, write the $15x^2$ as a linear combination of $\{1+2x+x^2,\ \ 1+2x-5x^2\}$. If not possible, state not possible.

$$\frac{5}{2}(1+2x+x^2)-\frac{5}{2}(1+2x-5x^2)$$

7e.) If $W = span\{1+2x+x^2,\ \ 1+2x-5x^2\}$, find a basis for $W^{\perp}$.

$3x - [\frac{9}{8} + \frac{9}{4}x - \frac{15}{8}x^2] = -\frac{9}{8} + \frac{3}{4}x + \frac{15}{8}x^2$

Hence a a basis for $W^{\perp}$ is $\{-\frac{9}{8} + \frac{3}{4}x + \frac{15}{8}x^2\}$.

Since $3x$ is not in W, $3x - proj_W(3x)$ is perpendicular to W and thus is in $W^{\perp}$. Since the orthogonal compliment of a 2-dimensional subspace of a 3-dimensional vector space is a 1-dimensional subspace (3 - 2 = 1), $W^{\perp}$ is 1-dimensional.

7f.) Find an orthogonal basis (according to the given inner product) for P_2 which contains the polynomials $1+2x+x^2$ and $1+2x-5x^2$

$$\{1+2x+x^2,\ 1+2x-5x^2,\ -\frac{9}{8} + \frac{3}{4}x + \frac{15}{8}x^2\}$$

This is what you would get by applying the G. S. algorithm, but note you have already done all the steps of G. S. in the preceding problems.

7g.) Is your answer to 5f an orthogonal basis for P_2 according to the inner product $< a_0, a_1x + a_2x^2, b_0 + b_1x + b_2x^2 > \ = \ a_0b_0 + a_1b_1 + a_2b_2$.

NO

Using the inner product

$< a_0, a_1x + a_2x^2, b_0 + b_1x + b_2x^2 > \ = \ a_0b_0 + a_1b_1 + a_2b_2$.

$< 1+2x+x^2, -\frac{9}{8} + \frac{3}{4}x + \frac{15}{8}x^2 >= -\frac{9}{8} + \frac{6}{4} + \frac{15}{8} \neq 0$.

Thus, the answer to 5f is NOT an orthogonal basis for P_2 according to the inner product $< a_0, a_1x + a_2x^2, b_0 + b_1x + b_2x^2 > \ = \ a_0b_0 + a_1b_1 + a_2b_2$.