1.3 Vectors in R^m

Defn: The vector $\mathbf{w}$ is a linear combination of the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ if there exist scalars $c_1, ..., c_n$ such that

$$\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n}$$

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

$$\begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix} \to \begin{bmatrix} 9 & 4 & 3 \\ 0 & \frac{44}{9} & -\frac{66}{9} \end{bmatrix} \to \begin{bmatrix} 9 & 4 & 3 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

$$\to \begin{bmatrix} 9 & 0 & 9 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

Thus, $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} - (3/2) \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

If possible, write $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix}$

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear comb'n of $\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -30 \\ 50 \end{bmatrix}$

If possible, write $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ as a l. c. of $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right\}$

If possible, write $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as a l. c. of $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & 1 \\ 2 & 5 & 7 & 3 & 0 \\ 3 & 6 & 9 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 & 0 & 1 \\ 0 & -3 & -3 & 3 & -2 \\ 0 & -6 & -6 & 6 & -3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 4 & 5 & 0 & 1 \\ 0 & -3 & -3 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 & 0 & * \\ 0 & 1 & 1 & -1 & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & 4 & * \\ 0 & 1 & 1 & -1 & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

$span\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ = the set of all linear combinations, $c_1\mathbf{v_1}+c_2\mathbf{v_2}+...+c_n\mathbf{v_n}$, of the vectors in $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ $\blacksquare$

= the hyperplane containing the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ anchored at $\mathbf{b} = \mathbf{0}$

= the hyperplane containing the points $\mathbf{0}, \mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$

Let $A = [\mathbf{a_1}...\mathbf{a_n}]$, where the a_i are k-vectors.

$\mathbf{b}$ is in $span\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ if and only if $Ax = b$ has at least one solution.

$span\{\mathbf{a_1}, ..., \mathbf{a_n}\} = R^k$ if and only if $Ax = b$ has at least one solution for every $\mathbf{b}$
(leading entry in every row).

Does $span\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\} = R^2$? Yes, since

$$x_1 \begin{bmatrix} 9 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ has a sol'n for all } \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

I.e., $\begin{bmatrix} 9 & 4 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has a sol'n for all $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Check:

$$\begin{bmatrix} 9 & 4 & b_1 \\ 7 & 8 & b_2 \end{bmatrix} \to \begin{bmatrix} 9 & 4 & b_1 \\ 0 & 8 - \frac{7}{9}(4) & b_2 - \frac{7}{9}(b_1) \end{bmatrix}$$

Thus solution exists no matter what b_1 and b_2 are.

Short-cut: $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$.

Thus span of $\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\}$ is 2-dimensional.

The only 2-dimensional plane in R^2 is R^2.

Note this short-cut only works in R^2

Does $span \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix} \right\} = R^2$?

Does $span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \\ 9 \end{bmatrix} \right\} = R^4$?

Does $span\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 10 \\ 4 \\ -3 \end{bmatrix} \} = R^3$?

$$\begin{bmatrix} 0 & 2 & 4 & 0 & 6 & 10 \\ 0 & 2 & 4 & -1 & 2 & 4 \\ 0 & -3 & -6 & 2 & -1 & -3 \end{bmatrix}$$

is row equivalent to

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Is $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ in the span of $\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\}$?

Yes, since $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = x_1 \begin{bmatrix} 9 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ has a solution.

Check:

$$\begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix} \to \begin{bmatrix} 9 & 4 & 3 \\ 0 & 8 - \frac{7}{9}(4) & -5 - \frac{7}{9}(3) \end{bmatrix}$$

Thus solution exists.

Short-cut: $span\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\} = R^2$

Is $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ in $span \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix} \right\}$?

Is $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ in $span \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix} \right\}$?

$$\text{Is } \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \text{ in } \mathit{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right\}?$$

$$\begin{bmatrix} 1 & 4 & 5 & 0 \\ 2 & 5 & 7 & 3 \\ 3 & 6 & 9 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -6 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \blacksquare$$

$$\text{Is } \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \text{ in } \mathit{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}?$$

$$\text{Is } \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \text{ in } \mathit{span} \left\{ \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}?$$

$$\text{Is } \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \text{ in } \mathit{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \right\}?$$

$$\text{Is } \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \text{ in } \mathit{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}?$$

$$\text{Is } \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \text{ in } \mathit{span} \left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right\}?$$