1.1, 1.2 Solving systems of linear equations.

Example: Solve

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 5y + 5z &= 4 \\ -x - 3y - 2z &= -4 \end{aligned}$$

$\downarrow$ eqn 2 - 2 eqn 1 $\rightarrow$ eqn 2,
$\downarrow$ eqn 3 + eqn 1 $\rightarrow$ eqn 3

$$\begin{aligned} x + 2y + 3z &= 0 \\ 0x + y - z &= 4 \\ 0x - y + z &= -4 \end{aligned}$$

$\downarrow$ eqn 3 + eqn 2 $\rightarrow$ eqn 3

$$\begin{aligned} x + 2y + 3z &= 0 \\ 0x + y - z &= 4 \\ 0x + 0y + 0z &= 0 \end{aligned}$$

$\downarrow$ eqn 1 - 2eqn 2 $\rightarrow$ eqn 1

$$\begin{aligned} x + 0y + 5z &= -8 \\ 0x + y - z &= 4 \end{aligned}$$

Thus $x = -8 - 5z$
$y = 4 + z$
$z = z$ (i.e., z is free, z can be any real number).

System of Linear Equations:

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ . \\ . \\ . \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{gathered}$$

Coefficient Matrix:

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ & & . & \\ & & . & \\ & & . & \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

Augmented Matrix Form:

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ & & . & & \\ & & . & & \\ & & . & & \\ a_{m1} & a_{m2} & ... & a_{mn} & b_m \end{bmatrix}$$

Elementary row operations:

$$R_i \to cR_i \text{ where } c \neq 0$$

$$R_i \leftrightarrow R_j$$

$$R_i \to R_i + cR_j$$

Two systems of equations are *equivalent* if they both have the same solution set.

If two augmented matrices are row-equivalent, the corresponding linear systems of equations are equivalent.

Methods of solving a system of linear equations:

1.) Put matrix in Echelon Form

2.) Put matrix in Reduced Echelon form

Echelon form (non-unique):

The leftmost nonzero element in each row is called a *leading entry* or *pivot*.

i.) In each column with a leading entry, all entries below the leading entry are zero.

ii.) Each leading entry of a row is to the left of the leading entry of any row below it.

iii.) All rows of all zeros are below all non-zero rows.

(Note in echelon form, I do not require that the leading entry equal 1)

The position of a leading entry is called the *pivot position*.

A *pivot column* is a column containing a leading entry.

The variable corresponding to a pivot column is called a *basic variable*.

Variables that do not correspond to a pivot column are called *free variables*.

Row-reduced echelon form (unique):

i.) The matrix is in echelon form.

ii.) The leading entries are all equal to 1.

iii.) In each column with a leading entry, all other entries are zero.

REQUIRED METHOD:

To put a matrix in echelon form work from left to right.

To put a matrix in row-reduced echelon form:

i.) First put in echelon form (work from left to right).

ii.) Put into reduced echelon form (work from right to left).

You may take short-cuts, but if your method is longer than the above, you will be penalized grade-wise.

Every matrix can be transformed by a finite sequence of elementary row operations into one that is in row-reduced echelon form.

Echelon form is not unique, but row-reduced echelon form is unique.

A system of LINEAR equations can have

i.) No solutions (inconsistent)

ii.) Exactly one solution

iii.) Infinite number of solutions.

To solve a system of equations:

i.) Create augmented matrix.

ii.) Put matrix into EF.

iii.) Put into REF.

iv.) Solve.

Case 1: If pivot in last column of augmented matrix.
Then system of equations has **no solution**.

Case 2: If no pivot in last column of augmented matrix:

a.) No free variables implies **unique solution**.

b.) Free variables imply an **infinite number of solutions**
Solve for pivot column variables in terms of free variables.

Solve:

$$
\begin{aligned}
3x + 6y + 9z &= 0 \\
4x + 5y + 6z &= 3 \\
7x + 8y + 9z &= 0
\end{aligned}
$$

1.5: A system of equations is **homogeneous** if $b_i = 0$ for all i.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{bmatrix}$$

A homogeneous system of LINEAR equations can have

a.) Exactly one solution ($\mathbf{x} = \mathbf{0}$)

b.) Infinite number of solutions (including, of course, $\mathbf{x} = \mathbf{0}$).

Solve:

$$\begin{aligned} 3x + 6y + 9z &= b_1 \\ 4x + 5y + 6z &= b_2 \\ 7x + 8y + 9z &= b_3 \end{aligned}$$

where 1a.) $b_1 = 0,\ b_2 = 0,\ b_3 = 0$

1b.) $b_1 = 0,\ b_2 = 3,\ b_3 = 0$

1c.) $b_1 = 6,\ b_2 = 5,\ b_3 = 8$

$$\begin{bmatrix} 3 & 6 & 9 & 0 & 0 & 6 \\ 4 & 5 & 6 & 0 & 3 & 5 \\ 7 & 8 & 9 & 0 & 0 & 8 \end{bmatrix}$$

$\downarrow \frac{1}{3}R_1 \to R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 4 & 5 & 6 & 0 & 3 & 5 \\ 7 & 8 & 9 & 0 & 0 & 8 \end{bmatrix}$$

$\downarrow R_2 - 4R_1 \to R_2, \quad R_3 - 7R_1 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & -6 & -12 & 0 & 0 & -6 \end{bmatrix}$$

$\downarrow R_3 - 2R_1 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & -6 & 0 \end{bmatrix}$$

$\downarrow$ already know sol'n to system b.

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & -3 & -6 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow -\frac{1}{3}R_2 \to R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 - 2R_2 \to R_1]{} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$