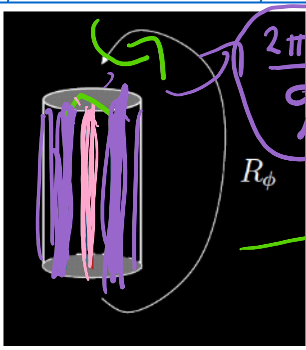
H. Seifert, *Topology of 3-dimensional fibered spaces*, A Textbook of Topology (H. Seifert, W. Threlfall) Translated by W. Heil from *Topologie dreidimensionales gefaserter Raum*, Acta Math., 60 (1933), 145–238.

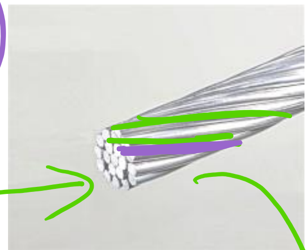
M. Jankins, W. D. Neumann, *Lectures on Seifert manifolds*, Brandeis Lecture Notes, 1983.

http://www.math.columbia.edu/~neumann/preprints/neumann_lectures%20on%20seifert%20manifolds.pdf

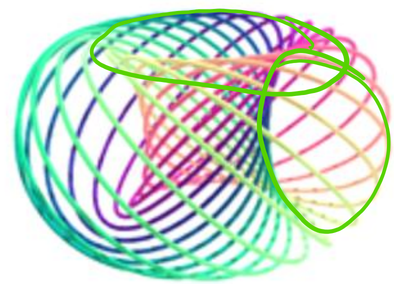


$\frac{2\pi p}{q}$

http://galileo.math.siu.edu/Courses/532/Sum13/



http://wire.buyawg.com/item/bare-aluminum-acsr/acsr-aluminum-conductor-steel-reinforced/bunting



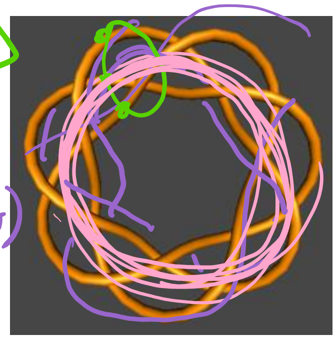
https://nilesjohnson.net/hopf.html

center line = exceptional fiber

$3L + 7M$

other fibers = ordinary or regular fibers

$\frac{7}{3}(2\pi)$



Knotplot

Any 2 ordinary fibers are isotopic

---

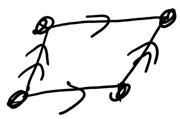
case 1: Everything orientable

$$M(g : (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$$

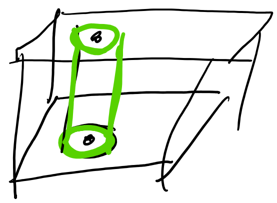
where S_g = orientable surface of genus g

$S_g \times S^1 \to$ trivial fibration

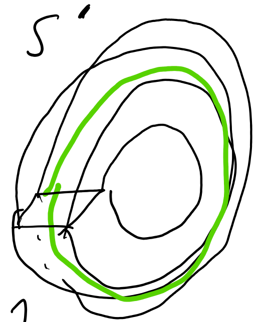
fibers = $pt \times S^1$



$\xrightarrow{S_g \times I}$



$\to$



$\{x\} \times 0 \sim \{x\} \times 1$

$$\left[S_g \times S^1 \setminus \bigcup_{i=1}^{n} (D^2 \times S^1) \right] \bigcup_{i=1}^{n} V^3 = M\left(g; (\alpha_i, \beta_i)_{i=1,\dots,n}\right)$$

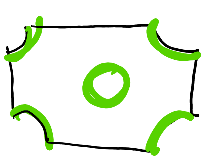
$V^3 = D^2 \times S^1$

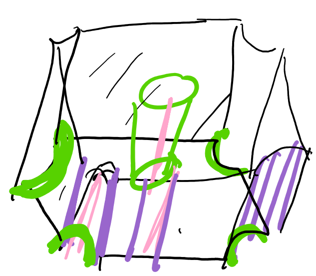
$\partial V^3 \to \partial D^2 \times S^1$

$m \to \alpha_i Q_i + \beta_i F$

Q_i = crossing curve $= \partial D_2 \times pt$

F = fiber

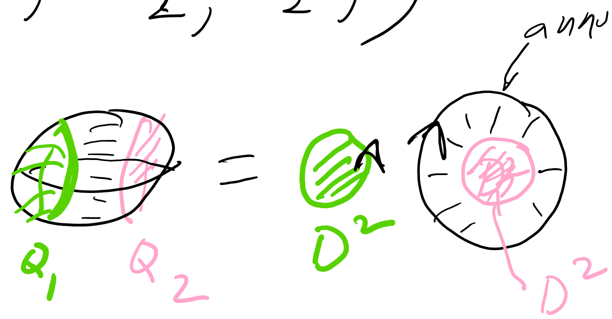




Ex = S_g = Torus
$g = 1$

$T_2 \setminus \left[(D^2 \times S^1) \cup (D^2 \times S^1) \right]$

$$M(0;\ (\alpha_1, \beta_1),\ (\alpha_2, \beta_2))$$

$g = 0 \Rightarrow S_g$ 

$$= D^2 \times S^1 \cup (\text{annulus} \times S^1) \cup D^2 \times S^1$$

$$= D^2 \times S^1 \cup D^2 \times S^1 = \text{Lens space}$$

$$M(0; (\alpha_1, \beta_1), (\alpha_2, \beta_2))$$

$$M_i \rightarrow \alpha_i Q_i + \beta_i F_i$$

$$L_i \rightarrow \alpha_i' Q_i + \beta_i' F_i$$

$$\begin{vmatrix} \alpha_i & \beta_i \\ \alpha_i' & \beta_i' \end{vmatrix} = 1$$

$$\begin{bmatrix} M_i \\ L_i \end{bmatrix} = \begin{bmatrix} \alpha_i & \beta_i \\ \alpha_i & \beta_i' \end{bmatrix} \begin{bmatrix} Q_i \\ F_i \end{bmatrix}$$

$$\begin{bmatrix} Q_i \\ F_i \end{bmatrix} = \underbrace{\begin{bmatrix} \beta_i' & -\beta_i \\ -\alpha_i' & \alpha_i \end{bmatrix} \begin{bmatrix} M_i \\ L_i \end{bmatrix}}$$

$Q_1 + Q_2 = \partial$ annulus $\times$ pt $\subset$ annulus $\times S^1$

$= 0$

 $\times S^1$

$Q_1 = -Q_2$

$F_1 = F_2$

$S^1 =$ Fibers

$$\begin{bmatrix} M_1 \\ L_1 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_1' & \beta_1' \end{bmatrix} \begin{bmatrix} Q_1 \\ F_1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_1' & \beta_1' \end{bmatrix} \begin{bmatrix} -Q_2 \\ F_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\alpha_1 & \beta_1 \\ -\alpha_1' & \beta_1' \end{bmatrix} \begin{bmatrix} Q_2 \\ F_2 \end{bmatrix}$$

$$\begin{bmatrix} -\alpha & \beta_1 \end{bmatrix} \begin{bmatrix} \beta_1' & -\beta \end{bmatrix} \begin{bmatrix} M_2 \end{bmatrix}$$

$$\searrow \quad = \begin{bmatrix} -\alpha_1 & \beta_1 \\ -\alpha_1' & \beta_1' \end{bmatrix} \begin{bmatrix} \beta_2' & -\beta_2 \\ -\alpha_2' & \alpha_2 \end{bmatrix} \begin{bmatrix} M_2 \\ L_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\alpha_1 \beta_2' - \beta_1 \alpha_2' & +\alpha_1 \beta_2 + \alpha_2 \beta_1 \\ \sim & \sim \end{bmatrix} \begin{bmatrix} M_2 \\ L_2 \end{bmatrix}$$

$$M \longrightarrow \underbrace{-(\alpha_1 \beta_2' + \beta_1 \alpha_2')}_{b} M_2 + \underbrace{(\alpha_1 \beta_2 + \alpha_2 \beta_1)}_{a} L_2$$

$\Rightarrow$ lens space $L(a, b)$ where

$$a = \det \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\beta_1 & \beta_2 \end{pmatrix}$$

$$b = -\det \begin{pmatrix} \alpha_1 & \alpha_2' \\ -\beta_1 & \beta_2' \end{pmatrix}$$

$$= M(\underset{\substack{\uparrow \\ \text{sphere}}}{O};\ (\alpha_1, \beta_1), (\alpha_2, \beta_2))$$

choice of α_1', β_1'

for any choice of α_2', β_2'

st $\det \begin{vmatrix} \alpha_2 & \alpha_2' \\ \beta_2 & \beta_2' \end{vmatrix} = 1$

A lens space is a SFS with base space S^2

Can also create some lens spaces w/ non orientable base space

---

SFS $\alpha_i \neq 0 \ \forall i$

GS = generalized SFS
allow $\alpha_i = 0$ for some i

SFS $\subset$ GS

$\frac{\beta_i}{\alpha_i} = \frac{-\beta_i}{-\alpha_i}$ Take $\alpha_i \geq 0$

2 GS are isomorphic via a fiber preserving map iff

1) Add or delete $(\alpha, \beta) = (1,0)$

2) Replace $(0, \pm 1)$ by $(0, \mp 1)$