To define a homology, one only needs

1.) Objects = basis for R-module, R[objects]
2.) Grading
3.) Boundary map

Note (1) is an unneeded restriction. More generally, one only needs a chain complex, a sequence of abelian groups (or modules) connected by homomorphisms ∂ such that $\partial^2 = 0$:

$$... \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} ...$$

To create long exact sequences (of pair, of triple, meyer-vietoris), one only needs appropriate short exact sequences.

Thus homology is algebra, not topology.

You don't need any topology to do homology.

A homology theory requires that homology respect certain aspects of the topology of a space.

Categories:

1.) Topological spaces w/morphisms = continuous maps $f : X \to Y$.

2.) Chain complexes with morphism = chain maps:

$$\begin{array}{ccccccc}
\cdots \to C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \to \cdots \\
\downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\
\cdots \to D_{n+1} & \xrightarrow{\delta_{n+1}} & D_n & \xrightarrow{\delta_n} & D_{n-1} \to \cdots
\end{array}$$

3.) $\bigoplus H_n(C)$ with morphism = homomorphism preserving grading.

Note we have covariant functors taking (1) to (2) to (3).

4.) $\bigoplus H^n(C)$ with morphism = homomorphism preserving grading.

Note we have a contravariant functor taking (1) to (4).

Example: The continuous map $f : X \to X \times X$, $f(x) = (x, x)$ induces a homomorphism $f^* : H^n(X \times X) \to H^n(X)$ by if $\phi : C_n(X \times X) \to R$, then $\phi \circ f : C_n(X) \to R$.

Thus in cohomology, we can create a product via the cross product (external cup product):

$p_1 : X \times Y \to X$.
$p_1^* : H^k(X) \to H^k(X \times Y)$, $p_1^*(\phi) = \phi \circ p_1 : C_k(X \times Y) \to R$

That is $p_1^*(k-$cocycle in $X) = k-$cocycle in $X \times Y$.

$$H^k(X) \times H^\ell(Y) \to H^{k+\ell}(X \times Y).$$

$$a \times b \to p_1^*(a) \smile p_2^*(b)$$

$$(p_1^*(a) \smile p_2^*(b))(\sigma) = p_1^*(a)(\sigma|_{[v_0,\ldots,v_k]}) \cdot p_2^*(b)(\sigma|_{[v_k,\ldots,v_{k+\ell}]}).$$

5.) $[X]$ such that $X_1 \sim X_2$ if there is a homotopy equivalence $h : X_1 \to X_2$ with morphisms $[f]$ where f is a continuous map and $f \sim g$ if f is homotopic to g.

Note H_n is a covariant functor taking (5) to (3), while H^n is a contravariant functor taking (5) to (4)

Defn: Let $(A, +)$ and $(B, +)$ be abelian groups. Then the tensor product is the abelian group $(A \otimes B, +)$ such that

1. There is a bilinear map $i\colon A \times B \to A \otimes B$ and

2. Given any bilinear map $f\colon A \times B \to C$, there is a unique linear map $L_f\colon A \otimes B \to C$ such that $L_f \circ i = f$.

Defn: $A \otimes B =< a \otimes b \mid (a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b,$
$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2 >$

Defn: Let R be a commutative ring and $(A, +)$ and $(B, +)$ be R-modules. Then the tensor product is the R-module $(A \otimes_R B, +)$ such that

1. There is a bilinear map $i\colon A \times B \to A \otimes_R B$ and

2. Given any bilinear map $f\colon A \times B \to C$, there is a unique linear map $L_f\colon A \otimes_R B \to C$ such that $L_f \circ i = f$.

Defn: $A \otimes_R B =< a \otimes_R b \mid (a_1 + a_2) \otimes_R b = a_1 \otimes_R b + a_2 \otimes_R b,$
$a \otimes_R (b_1 + b_2) = a \otimes_R b_1 + a \otimes_R b_2, ra \otimes_R b = a \otimes_R rb >$

Ex: If R has an identity 1, $R \otimes_R A \cong R$.

$\phi(r_1 \otimes a) = r_1 a$. with inverse $\psi : R \to R \otimes_R A$, $\psi(a) = 1 \otimes a$.

$\phi(\psi(a)) = \phi(1 \otimes a) = a.$

$\psi(\phi(r \otimes a) = \psi(ra) = 1 \otimes ra = r \otimes a.$

Useful facts:

1.) $A \otimes B \cong B \otimes A$.

2.) $(\bigoplus_i A_i) \otimes B \cong \bigoplus_i (A_i \otimes B)$.

3.) $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

4.) $\mathbb{Z} \otimes A \cong A$ via isomorphism $\phi(n \otimes a) = na$.

with inverse $\psi : A \to \mathbb{Z} \otimes A$, $\psi(a) = 1 \otimes a$.

$$\phi(\psi(a)) = \phi(1 \otimes a) = a.$$

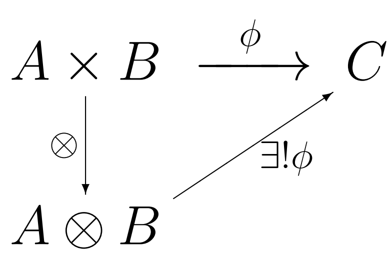
$$\psi(\phi(n \otimes a) = \psi(na) = 1 \otimes na = n \otimes a.$$

5.) $\mathbb{Z}_n \otimes A \cong A/nA$ via the isomorphism $\phi(n \otimes a) = na$.

Ex: $\mathbb{Z}_n \otimes \mathbb{Z} \cong Z_n$ while $\mathbb{Z}_n \otimes \mathbb{Q} \cong 0$

6.) Homomorphisms $f_i : A_i \to B_i$ induce a homomorphism $f_1 \otimes f_2 : A_1 \otimes A_2 \to B_1 \otimes B_2$, $(f_1 \otimes f_2)(a_1 \otimes a_2) = f_1(a_1) \otimes f_2(a_2)$.

7.) A bilinear map $\phi : A \times B \to C$ induces a homomorphism $A \otimes B \to C$, sending $a \otimes b$ to $\phi(a, b)$.



8.) $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact implies

$$A \otimes G \xrightarrow{f \otimes i_G} B \otimes G \xrightarrow{g \otimes i_G} C \otimes G \to 0 \text{ exact}$$

$$\begin{array}{ccc} H^*(X;R) \times H^*(Y;R) & \xrightarrow{\times} & H^*(X \times Y;R) \\ \Big\downarrow \otimes & \nearrow_{\exists! \phi} & \\ H^*(X;R) \otimes H^*(X;R) & & \end{array}$$

$\mu(a \otimes b) = a \times b$

Let $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$ where $|x|$ = dimension of x.

$$\mu((a \otimes b)(c \otimes d)) = (-1)^{|b||c|} \mu(ac \otimes bd) = (-1)^{|b||c|} (a \smile c) \times (b \smile d)$$

$$= (-1)^{|b||c|} p_1^*(a \smile c) \smile p_2^*(b \smile d)$$

$$= (-1)^{|b||c|} p_1^*(a) \smile p_1^*(c) \smile p_2^*(b) \smile p_2^*(d)$$

$$= (-1)^{|b||c|} (-1)^{|b||c|} p_1^*(a) \smile p_2^*(b) \smile p_1^*(c) \smile p_2^*(d)$$

$$= p_1^*(a) \smile p_2^*(b) \smile p_1^*(c) \smile p_2^*(d)$$

Theorem 1 (Künneth formula)**.** *The cross product $H^\bullet(X;R) \otimes_R H^\bullet(Y;R) \to H^\bullet(X \times Y;R)$ is an isomorphism of rings if X and Y are CW complexes and $H^k(Y;R)$ is a finitely generated free R-module for all k.*

The hypothesis X and Y are CW complexes is unnecessary. The result also hold in a relative setting.

Theorem 2 (Relative Künneth formula)**.** *For CW pairs* (X, A) *and* (Y, B) *the cross product homomorphism*
$H^\bullet(X, A; R) \otimes_R H^\bullet(Y, B; R) \to H^\bullet(X \times Y, A \times Y \cup X \times B; R)$
is an isomorphism of rings if $H^k(Y, B; R)$ *is a finitely generated free* R*-module for each* k*.*

The General Künneth Formula

Theorem 3 (Künneth formula for PID)**.** *If* X *and* Y *are CW complexes and* R *is a principal ideal domain, then there are split short exact sequences*

$$0 \to \bigoplus_i H_i(X; R) \otimes_R H_{n-i}(Y; R)$$

$$\to H_n(X \times Y; R) \to \bigoplus_i \operatorname{Tor}_R(H_i(X; R), H_{n-i-1}(Y; R)) \to 0$$

natural in X *and* Y*.*

Corollary 1. *If* F *is a field and* X *and* Y *are CW complexes, then the cross product map*

$$h\colon \bigoplus_i H_i(X; F) \otimes_F H_{n-i}(Y; F) \to H_n(X \times Y; F)$$

is an isomorphism for all n*.*

Universal Coefficients for Homology

Theorem 4 (Universal Coefficients for Homology)**.** *For each pair of spaces* (X, A) *there are split exact sequences*

$$0 \longrightarrow H_n(X, A) \otimes G \longrightarrow H_n(X, A; G) \longrightarrow \operatorname{Tor}(H_{n-1}(X, A), G) \longrightarrow 0$$

for all n*, and these sequences are natural with respect to maps* $(X, A) \to (Y, B)$*.*

The following result enables us to compute the Tor groups.

Proposition 1.

1. *$\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$.*
2. *$\mathrm{Tor}(\bigoplus_i A_i, B) \cong \bigoplus_i \mathrm{Tor}(A_i, B)$.*
3. *$\mathrm{Tor}(A, B) = 0$ if A or B is free, or more generally torsion-free.*
4. *$\mathrm{Tor}(A, B) \cong \mathrm{Tor}(T(A), B)$ where $T(A)$ is the torsion subgroup of A.*
5. *$\mathrm{Tor}(\mathbb{Z}/n, A) \cong \mathrm{Ker}(A \xrightarrow{n\cdot} A)$.*
6. *For each short exact sequence $0 \to B \to C \to D \to 0$ there is a natural associated exact sequence*

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \to A \otimes B \to A \otimes C \to A \otimes$$

Corollary 2.

1. *$H_n(X; \mathbb{Q}) \cong H_n(X; \mathbb{Z}) \otimes \mathbb{Q}$, so when $H_n(X; \mathbb{Z})$ is finitely generated, the dimension of $H_n(X; \mathbb{Q})$ as a $\mathbb{Q}$-vector space equals the rank of of $H_n(X; \mathbb{Z})$.*
2. *If $H_n(X; \mathbb{Z})$ and $H_{n-1}(X; \mathbb{Z})$ are finitely generated, then for p prime, $H_n(X; \mathbb{Z}/p)$ consists of*
   (a) *a $\mathbb{Z}/p$ summand for each $\mathbb{Z}$ summand of $H_n(X; \mathbb{Z})$,*
   (b) *a $\mathbb{Z}/p$ summand for each $\mathbb{Z}/p^k$ summand in $H_n(X; \mathbb{Z})$, $k \geq 1$,*
   (c) *a $\mathbb{Z}/p$ summand for each $\mathbb{Z}/p^k$ summand in $H_{n-1}(X; \mathbb{Z})$, $k \geq 1$.*

Corollary 3.

1. *$\widetilde{H}_\bullet(X;\mathbb{Z}) = 0$ if and only if $\widetilde{H}_\bullet(X;\mathbb{Q}) = 0$ and $\widetilde{H}_\bullet(X;\mathbb{Z}/p) = 0$ for all primes p.*

2. *A map $f\colon X \to Y$ induces isomorphisms on homology with $\mathbb{Z}$ coefficients if and only if it induces isomorphisms on homology with $\mathbb{Q}$ and $\mathbb{Z}/p$ coefficients for all primes p.*

Theorem 5 (Alexander Duality)**.** *If K is a compact, locally contractible, non-empty, proper subspace of S^n, then*

$$\widetilde{H}_i(S^n \setminus K) \cong \widetilde{H}^{n-i-1}(K) \text{ for all } i.$$