Let $A \subset X$. $C_n(X, A) = C_n(X)/C_n(A)$, the quotient group (algebraic, not topological).

$\alpha \in C_n(A)$ implies $\partial\alpha \in C_{n-1}(A)$ and thus
$\partial_X : C_n(X) \to C_{n-1}(X)$ induces a map $\partial : C_n(X, A) \to C_{n-1}(X, A)$

$Z_n(X, A) =$ set of relative cycles: $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$.

$B_n(X, A) =$ set of relative boundaries: $\alpha \in C_n(X)$ such that
$\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

Thus as usual $H_n(X, A) = ker\partial_n / im\partial_{n+1} = Z_n(X, A)/B_n(X, A)$

where $C_{n+1}(X, A) \xrightarrow{\partial_{n+1}} C_n(X, A) \xrightarrow{\partial_n} C_{n-1}(X, A)$

Ex: $(\mathbb{R}, \{0, 1\})$

$H_1(\mathbb{R}) = 0, \quad H_1(\{0, 1\}) = 0, \quad H_1(\mathbb{R}, \{0, 1\}) = \mathbb{Z}$

Obvious Lemma: $0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{q} C_n(X, A) \to 0$ is a short exact sequence.

Pf: The inclusion map, i, is injective. The (algebraic) quotient map is surjective.
The image of $C_n(A)$ in $C_n(X)$ is sent to $[0]$ in $C_n(X, A)$.

Thus this short exact sequence induces a long exact sequence:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{\pi_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \longrightarrow \cdots .$$

If $[\sigma] \in H_n(X,A)$, then $\sigma \in C_n(X)$ and $\partial\sigma \in C_{n-1}(A)$

That is $\partial_*([\sigma]) = \partial_X(\sigma)$.

$$\begin{array}{ccccc} & & C_n(X) & \xrightarrow[\pi_n]{} & C_n(X,A) \\ & & \downarrow{\scriptstyle \partial_X} & & \\ C_{n-1}(A) & \xrightarrow[i_{n-1}]{} & C_{n-1}(X) & & \end{array}$$

Ex:

$$\cdots \longrightarrow H_1(\{0,1\}) \xrightarrow{i_*} H_1(\mathbb{R}) \xrightarrow{\pi_*} H_1(\mathbb{R},\{0,1\}) \xrightarrow{\partial_*} \widetilde{H_0}(\{0,1\}) \xrightarrow{i_*} \widetilde{H_0}(\mathbb{R}) \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \xrightarrow{i_*} 0 \xrightarrow{\pi_*} H_1(\mathbb{R},\{0,1\}) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{i_*} 0 \longrightarrow \cdots$$

Thus $H_1(\mathbb{R},\{0,1\}) = \mathbb{Z}$