Math 6400 Quiz October 7, 2018

1a.) Give a cellular decomposition of $S^2 \vee S^4$

Let v = a point = 0-dimensional simplex = 0-dimensional cellular complex

Let Δ^n = n-dimensional ball $\cong$ n-dimensional simplex

$$v \bigcup_{\partial\Delta^2 = v} \Delta^2 \bigcup_{\partial\Delta^4 = v} \Delta^4$$

The chain complex, $\cdots \to C_n \to \cdots$ for this decomposition is

$$\cdots \to 0 \to 0 \to C_4 =< \Delta^4 >= \mathbb{Z} \to 0 \to C_2 =< \Delta^2 >= \mathbb{Z} \to 0 \to C_0 =< v >= \mathbb{Z} \to 0$$

I.e., $\cdots \xrightarrow{\partial_7} 0 \xrightarrow{\partial_6} 0 \xrightarrow{\partial_5} \mathbb{Z} \xrightarrow{\partial_4} 0 \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} 0 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$

1b.) Use this cellular decomposition of $S^2 \vee S^4$ to calculate $H_n(S^2 \vee S^4)$

$$H_n(S^2 \vee S^4) = ker\partial_n / im\partial_{n+1}$$

$$ker\partial_n = \begin{cases} \mathbb{Z} & n = 0, 2, 4 \\ 0 & else \end{cases}$$

$im\partial_{n+1} = 0 \quad \forall n$ Thus $H_n(S^2 \vee S^4) = \begin{cases} \mathbb{Z} & n = 0, 2, 4 \\ 0 & else \end{cases}$

1c.) Use the universal coefficient theorem for cohomology to calculate $H^n(S^2 \vee S^4)$

$$H^n(S^2 \vee S^4; R) \cong Hom(H_n(S^2 \vee S^4), R = \begin{cases} Hom(\mathbb{Z}, R) & n = 0, 2, 4 \\ Hom(0, R) & else \end{cases} = \begin{cases} R & n = 0, 2, 4 \\ 0 & else \end{cases}$$ since

$$Hom(0, R) = \{f : \{0\} \to R, f(0) = 0\}$$

$$Hom(\mathbb{Z}, R) = \{f : \mathbb{Z} \to R, f(k) = kr \text{ where } f(1) = r\} = R$$

Note homomorphisms, f with domain $\mathbb{Z}$ are defined by the value $f(1)$. Thus we have an isomorphism $E : Hom(\mathbb{Z}, R) \to R$, $E(f) = f(1)$.

1d.) Use the definition of cohomology and the decomposition from 1a to calculate $H^n(S^2 \vee S^4; \mathbb{Z})$

By definition, $C^n(S^2 \vee S^4; \mathbb{Z}) = Hom(C_n \to \mathbb{Z}) = \begin{cases} Hom(\mathbb{Z}, \mathbb{Z}) & n = 0, 2, 4 \\ Hom(0, R) & else \end{cases} = \begin{cases} \mathbb{Z}, & n = 0, 2, 4 \\ 0 & else \end{cases}$

The chain complex is

$$\cdots \xleftarrow{\delta_7} 0 \xleftarrow{\delta_6} 0 \xleftarrow{\delta_5} \mathbb{Z} \xleftarrow{\delta_4} 0 \xleftarrow{\delta_3} \mathbb{Z} \xleftarrow{\delta_2} 0 \xleftarrow{\delta_1} \mathbb{Z} \xleftarrow{\delta_0} 0$$

$$ker\delta_n = \begin{cases} \mathbb{Z} & n = 1, 3, 5 \\ 0 & else \end{cases}$$

$im\delta_{n+1} = 0 \quad \forall n$ Thus $H_n(S^2 \vee S^4) = ker\delta_{n+1}/im\delta_n = \begin{cases} \mathbb{Z} & n = 0, 2, 4 \\ 0 & else \end{cases}$

1e.) If $\phi, \psi \in \bigoplus_n H^n(S^2 \vee S^4; \mathbb{Z})$, then $\phi \smile \psi = \begin{cases} k\phi & if \psi = k\mathbb{1} \in H^0(S^2 \vee S^4; \mathbb{Z}) \\ m\psi & if \phi = m\mathbb{1} \in H^0(S^2 \vee S^4; \mathbb{Z}) \\ 0 & else \end{cases}$