Steenrod five-lemma: Given the following commutative diagram of abelian groups where the horizontal sequence are exact, show that if f_1, f_2, f_4, f_5 are isomorphisms, so is f_3.

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Claim 1: f_2, f_4 onto and f_5 1:1, then f_3 onto.

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Claim 2: f_2, f_4 1:1 and f_1 onto, then f_3 1:1

$$\begin{CD}
A @>>> B @>>> C @>>> D @>>> E \\
@V{f_1}VV @V{f_2}VV @V{f_3}VV @V{f_4}VV @V{f_5}VV \\
A' @>>> B' @>>> C' @>>> D' @>>> E'
\end{CD}$$

$$\begin{CD}
A @>>> B @>>> C @>>> D @>>> E \\
@V{f_1}VV @V{f_2}VV @V{f_3}VV @V{f_4}VV @V{f_5}VV \\
A' @>>> B' @>>> C' @>>> D' @>>> E'
\end{CD}$$

$$\begin{CD}
A @>>> B @>>> C @>>> D @>>> E \\
@V{f_1}VV @V{f_2}VV @V{f_3}VV @V{f_4}VV @V{f_5}VV \\
A' @>>> B' @>>> C' @>>> D' @>>> E'
\end{CD}$$