

Obvious Lemma: $0 \rightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{q} C_n(X, A) \rightarrow 0$ is a short exact sequence.

Thus this short exact sequence induces a long exact sequence:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \longrightarrow \cdots$$

Example: If $A = \{p\}$, then

$$\begin{aligned} \cdots \longrightarrow H_n(\{p\}) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X, \{p\}) \xrightarrow{\partial_*} H_{n-1}(\{p\}) \longrightarrow \cdots \\ \cdots \longrightarrow 0 \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X, \{p\}) \xrightarrow{\partial_*} 0 \longrightarrow \cdots \end{aligned}$$

Thus $H_n(X, \{p\}) = H_n(X)$ for $n > 0$.

By defn of H_0 , $H_0(X, \{p\}) = \widetilde{H}_0(X)$.

Thus $H_n(X, \{p\}) = \widetilde{H}_n(X)$ for all n .

If $A \subset B \subset X$, then $0 \rightarrow C_n(B, A) \xrightarrow{i} C_n(X, A) \xrightarrow{q} C_n(X, B) \rightarrow 0$ is a short exact sequence.

Thus this short exact sequence induces a long exact sequence of the triple (X, B, A) :

$$\cdots \longrightarrow H_n(B, A) \xrightarrow{i_*} H_n(X, A) \xrightarrow{q_*} H_n(X, B) \xrightarrow{\partial_*} H_{n-1}(B, A) \longrightarrow \cdots$$

Note if A is a deformation retract of B , then $H_k(B, A) = 0$ for all k and $H_n(X, A) \cong H_n(X, B)$.

Thm 2.20 (Excision Theorem).

1.) If $\bar{Z} \subset A^\circ \subset X$, then $i : (X - Z, A - Z) \hookrightarrow (X, A)$

induces isomorphism: $H_n(X - Z, A - Z) \cong H_n(X, A)$

2.) If $X = A^\circ \cup B^\circ$, then $i : (B, A \cap B) \hookrightarrow (X, A)$

induces isomorphism: $H_n(B, A \cap B) \cong H_n(X, A)$

Note (1) and (2) are equivalent (let $B = X - Z$).

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Defn: The pair (X, A) is **good** if $A \neq \emptyset$ and $\exists V$ such that $\bar{A} \subset V^\circ \subset X$ and V deformation retracts onto A .

Thm: If (X, A) is good, then $H_n(X, A) \cong \widetilde{H}_n(X/A)$.

Proof:

$$\begin{array}{ccc}
 H_n(X, A) & \xrightarrow[\text{LES of triple}]{\cong} & H_n(X, V) & \xleftarrow[\text{excision}]{\cong} & H_n(X - A, V - A) \\
 \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\
 H_n(X/A, A/A) & \xrightarrow[\text{LES of triple}]{\cong} & H_n(X/A, V/A) & \xleftarrow[\text{excision}]{\cong} & H_n(X/A - A/A, V/A - A/A)
 \end{array}$$

