$H_1(T^2;\mathbb{Z}) \cong \mathbb{Z}\times\mathbb{Z}$ with basis $i=1,2$, $\phi_i(e_j) = \begin{cases} 1 & j=i,3 \\ 0 & else \end{cases}$

Note: in terms of basis for C^1, $\quad \phi_i = \psi_{e_1} + \psi_{e_3}, \quad$ for $i=1,2$.

Note: $(\delta\circ\phi_1)(\sigma_i) = \phi_1(\partial(\sigma_i)) = \phi_1(e_1+e_2-e_3)$

$$= \phi(e_1) + \phi(e_2) - \phi_(e_3) = 1+0-1 = 0$$

$(\phi_1 \smile \phi_1)(\sigma_1) = \phi_1(e_1)\cdot\phi_1(e_2) = (1)(0) = 0$

$(\phi_1 \smile \phi_1)(\sigma_2) = \phi_1(e_2)\cdot\phi_1(e_1) = (0)(1) = 0$

Thus $\phi_1 \smile \phi_1 = 0$. Similarly, $\phi_2 \smile \phi_2 = 0$.

$(\phi_1 \smile \phi_2)(\sigma_1) = \phi_1(e_1)\cdot\phi_2(e_2) = (1)(1) = 1$

$(\phi_1 \smile \phi_2)(\sigma_2) = \phi_1(e_2)\cdot\phi_2(e_1) = (0)(0) = 0$

$(\phi_2 \smile \phi_1)(\sigma_1) = \phi_2(e_1)\cdot\phi_1(e_2) = (0)(0) = 0$

$(\phi_2 \smile \phi_1)(\sigma_2) = \phi_2(e_2)\cdot\phi_1(e_1) = (1)(1) = 1$

Thus $\phi_1 \smile \phi_2 \neq \phi_2 \smile \phi_1$ in C^2.

$H_2(T^2;\mathbb{Z}) \cong \mathbb{Z}$ since $C^3 = 0$ implies $ker\delta = C^2$ and

$\delta\circ\psi_{e_i}(\sigma_j) = \psi_{e_i}(\partial(\sigma_j)) = \psi_{e_i}(e_1+e_2-e_3) = \begin{cases} 1 & i=1,2 \\ -1 & i=3 \end{cases}$

Thus $\delta\circ\psi_{e_i} = \psi_{\sigma_1} + \psi_{\sigma_2}$ for $i=1,2$ and $\delta\circ\psi_{e_3} = -\delta\circ\psi_{e_1}$

Thus $im\delta =< \psi_{\sigma_1} + \psi_{\sigma_2} >$. Thus in H^2, $\psi_{\sigma_1} = -\psi_{\sigma_2}$