Recall $\prod_n H^n(X;G)$ is an abelian group under addition.

$([\phi_0], [\phi_1], [\phi_2], ...) \in \prod_n H^n(X;G)$

where $\phi_n : C_n \to G \ \in \ ker(\delta) \ = \ Z_n^*(X;G) \ \subset \ C^n(X;G)$

$\phi \sim \psi \in H^n(X;G)$ iff $\phi = \psi + \delta\chi$ for some $\chi \in C^{n+1}(X;G)$.

Defn: A is a **graded ring** if $A = \bigoplus_{k\geq 0} A_k$ where
A_k are additive groups with multiplication $m : A_k \times A_\ell \to A_{k+\ell}$

Defn: $a_k \in A_k$ has **grade = degree = dimension** $= |a_k| = k$

Examples: Polynomial rings.

$R[x] = \{r_0 + r_1 x + ... + r_k x^k \mid r_i \in R, k \in \mathbb{N}\}$

$R[x]/(x^n) = \{r_0 + r_1 x + ... + r_k x^k \mid r_i \in R, k \in \{0, ..., n-1\}\}$
where $x^i x^j = x^m$ where $m = i + j \ mod \ n$.

$R[x_1, ..., x_n]$ where $r x_1^{i_1} x_2^{i_2} ... x_\ell^{i_n}$ has grade $\sum_{j=1}^n i_j$

Example: Exterior Algebras $\Lambda_R[x_1, ..., x_n]$

Basis for grade k: $x_{i_1} x_{i_2} ... x_{i_k}$ where $i_1 < i_2 < ... < i_k$

and $x_i x_j = -x_j x_i$ and $x_i^2 = 1$ for all i, j

Cup Product Let R be a commutative ring with identity.

Defn: For cochains $\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$,
the **cup product** $\phi \smile \psi \in C^{k+\ell}(X;R)$ is the cochain defined by

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|_{[v_0,...,v_k]}) \cdot \psi(\sigma|_{[v_k,...,v_{k+\ell}]}).$$

where $\sigma \in C_{k+\ell}(X)$.

The cup product is associative:

$\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, $\chi \in C^m(X;R)$

$$(\chi \smile (\phi \smile \psi))(\sigma) = \chi(\sigma|_{[v_0,...,v_m]}) \cdot (\phi \smile \psi)(\sigma|_{[v_m,...,v_{m+k+\ell}]})$$

$$= \chi(\sigma|_{[v_0,...,v_m]}) \cdot \phi(\sigma|_{[v_m,...,v_{m+k}]}) \cdot \psi(\sigma|_{[v_{m+k},...,v_{m+k+\ell}]})$$

$$= (\chi \smile \phi)(\sigma|_{[v_0,...,v_{m+k}]}) \cdot \psi(\sigma|_{[v_{m+k},...,v_{m+k+\ell}]})$$

$$= ((\chi \smile \phi) \smile \psi)(\sigma)$$

The cup product is distributive: If $k = \ell$

$$(\chi \smile (\phi + \psi))(\sigma) = \chi(\sigma|_{[v_0,...,v_m]}) \cdot (\phi + \psi)(\sigma|_{[v_m,...,v_{m+k}]})$$

$$= \chi(\sigma|_{[v_0,...,v_m]}) \cdot [\phi(\sigma|_{[v_m,...,v_{m+k}]}) + \psi(\sigma|_{[v_m,...,v_{m+k}]})]$$

$$= \chi(\sigma|_{[v_0,...,v_m]}) \cdot \phi(\sigma|_{[v_m,...,v_{m+k}]}) + \chi(\sigma|_{[v_0,...,v_m]}) \cdot \psi(\sigma|_{[v_m,...,v_{m+k}]})$$

$$= (\chi \smile \phi)(\sigma) + (\chi \smile \psi)(\sigma)$$

Similarly $((\phi + \psi) \smile \chi)(\sigma) = (\phi \smile \chi)(\sigma) + (\psi \smile \chi)(\sigma)$

Thus if R is a ring, $\bigoplus_n C^n(X;R)$ is a ring.

If R has a multiplicative identity, 1, then $\iota : C_0(X) \to R$, $\iota(v) = 1$ for all vertices v is the multiplicative identity for $\bigoplus_n C^n(X;G)$:

$$(\iota \smile \psi)(\sigma) = \iota(\sigma|_{[v_0]}) \cdot \psi(\sigma|_{[v_0,\dots,v_\ell]}) = 1 \cdot \psi(\sigma|_{[v_0,\dots,v_\ell]}) = \psi(\sigma)$$

$$(\phi \smile \iota)(\sigma) = \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \iota(\sigma|_{[v_k]}) = \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot 1 = \phi(\sigma).$$

Proposition 1. *For $\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, we have*

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi.$$

Note: The cup product of two cocycles is a cocycle.

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi = 0 + 0 = 0.$$

The cup product of a cocycle and a coboundary is a coboundary.

$$(-1)^k \phi \smile \delta\psi = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi = \delta(\phi \smile \psi).$$

The cup product of a coboundary and a cocycle is a coboundary.

$$\delta\phi \smile \psi = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi = \delta(\phi \smile \psi).$$

Proof: $\delta(\phi \smile \psi),\ \ \phi \smile \delta(\psi),\ \ \delta\phi \smile \psi \ \in \ C^{k+\ell+1}(X;R)$,

$$\delta\phi \in C^{k+1}(X;R) \text{ and } \delta\psi \in C^{\ell+1}(X;R)$$

$$(\delta\phi \smile \psi)(\sigma) = \phi(\partial(\sigma|_{[v_0,\dots,v_{k+1}]})) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$

$$= \sum_{i=0}^{k+1} (-1)^i \phi(\sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}).$$

$(\phi \smile \delta\psi)(\sigma)$

$$= \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\partial(\sigma|_{[v_k,\dots,v_{k+\ell+1}]})) = \sum_{i=k}^{k+\ell+1} (-1)^{i-k} \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,\widehat{v_i},\dots,v_{k+\ell+1}]})$$

Thus $[\delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi](\sigma)$

$$= \sum_{i=0}^{k+1} (-1)^i \phi(\sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$

$$+(-1)^k \sum_{i=k}^{k+\ell+1} (-1)^{i-k} \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,\widehat{v_i},\dots,v_{k+\ell+1}]})$$

$$= \sum_{i=0}^{k} (-1)^i \phi(\sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}) + (-1)^{k+1} \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$

$$+(-1)^k \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}) + \sum_{i=k+1}^{k+\ell+1} (-1)^i \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,\widehat{v_i},\dots,v_{k+\ell+1}]})$$

$$= \sum_{i=0}^{k} (-1)^i \phi(\sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}) + \sum_{i=k+1}^{k+\ell+1} (-1)^i \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,\widehat{v_i},\dots,v_{k+\ell+1}]})$$

$$= \sum_{i=0}^{k+\ell+1} (-1)^i (\phi \smile \psi)(\sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_{k+\ell+1}]}) = (\phi \smile \psi)(\partial(\sigma)) = \delta(\phi \smile \psi)(\sigma)$$