$Hom(A, G) = \{h : A \to G \mid h \text{ homomorphism }\}$

$Hom(A, G)$ is a group under function addition.

The **dual homomorphism to** $f : A \to B$ is the

homomorphism $f^* : Hom(A, G) \leftarrow Hom(B, G)$

defined by $f^*(\psi) = \psi \circ f : A \to B \to G$

That is the assignment

$$A \to Hom(A, G) \quad \text{and} \quad f \to f^*$$

is a **contravarient functor** from the category of abelian groups and homomorphisms to itself since

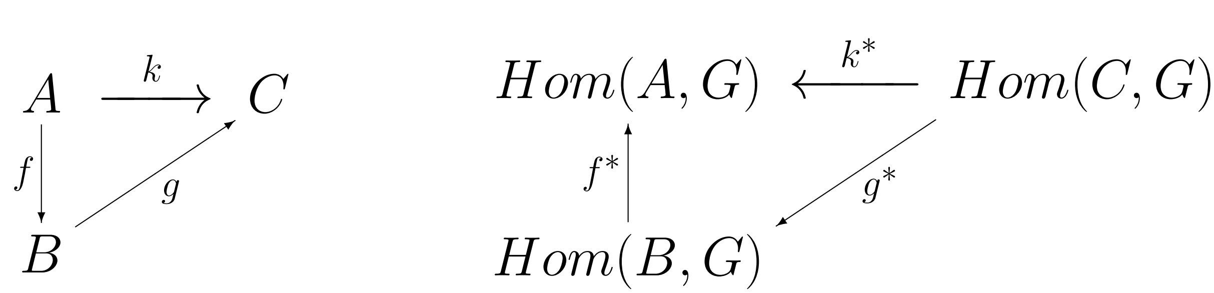
If $i : A \to A$ is the identity map on A, then

$$i_*(\psi) = \psi \circ i = \psi \text{ is the identity map on } Hom(A, G).$$

And if $f : A \to B, \quad g : B \to C, \quad \psi : C \to G$

$$(f^* \circ g^*)(\psi) = f^*(g^*(\psi)) = f^*(\psi \circ g) = \psi \circ g \circ f = (g \circ f)^*(\psi)$$

In other words, if the diagram on the left commutes, so does the one on the right:



- Hence f isomorphism implies f^* is an isomorphism.

- The constant fn $f = 0$ implies $f^* = 0$ since $f^*(\psi) = \psi \circ f = \psi \circ 0$.

Given a chain complex:

$$... \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial n} C_{n-1} \to ...$$

Its dual is also a chain complex:

$$... \leftarrow Hom(C_{n+1}, G) \xleftarrow{\partial^*_{n+1}} Hom(C_n, G) \xleftarrow{\partial^*_n} Hom(C_{n-1}, G) \leftarrow ...$$

Cohomology

Cochains: $\Delta^n(X; G) = Hom(C_n, G) = \prod_{\sigma_\alpha} G$

Coboundary map: $\delta^1 = \partial^*_1 : \Delta^0(X; G) \to \Delta^1(X; G)$

Cohomology: $H^n(X; G) = Z^n(X; G)/B^n(X; G) = ker(\delta_{n+1})/im(\delta_n)$

$\underline{n = 0}$:

The dual of $C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$ is $\Delta^1(X; G) \xleftarrow{\delta_1} \Delta^0(X; G) \xleftarrow{\delta_0} 0$

$im(\delta_0) = 0$. Thus $H^0(X; G) = ker(\delta_1)/im(\delta_0) = ker(\delta_1)$

$\psi : C_0 \;=\; < V > \;\longrightarrow G,$ defined by $\psi(v_\alpha) = g_\alpha$

$\delta_1(\psi) : C_1 \;=\; < E > \;\longrightarrow G,$

$\delta_1(\psi)([v_1, v_2]) = \psi \circ \delta([v_1, v_2]) = \psi(v_2 - v_1) = \psi(v_2) - \psi(v_1).$

Application: ψ = elevation, $\delta_1(\psi)$ = change in elevation.
Application:
ψ = voltage at connection points, $\delta_1(\psi)$ = voltage across components.

$\delta_1(\psi) = 0$ iff

$\delta_1(\psi)([v_1, v_2]) = \psi \circ \delta_1([v_1, v_2]) = \psi(v_2 - v_1) = \psi(v_2) - \psi(v_1) = 0.$

Thus

$ker(\delta_1) = \{\psi : C_0 \to G \mid \psi \text{ is constant on the components of } X\}$

$$\text{Hence } H^0(X; G) = \prod_{\text{components of } X} G.$$

Recall $H_0(X; G) = \sum_{\text{components of } X} G.$

$\underline{n = 1}$:

Dual of $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ is $\Delta^2(X; G) \xleftarrow{\delta_2} \Delta^1(X; G) \xleftarrow{\delta_1} \Delta^0(X; G)$

$H^1(X; G) = Z^1(X; G)/B^1(X; G) = ker(\delta_2)/im(\delta_1)$

$im(\delta_1) = ?$

Suppose $\delta_1(\psi) = \sigma : \Delta^1 \to G$

Then σ is determined by trees in the 1-skeleton of $X = X^1$.

Let $T =$ a set of maximal trees for X^1 & let $A = \{e_a \in \Delta^1 \mid e_a \notin T\}$.

If $\Delta^2 = 0$, $H^1(X; G) = ker(\delta_2)/im(\delta_1) = \Delta^1/im(\delta_1) = \prod_{e_\alpha \in A} G$

Recall if $\Delta^2 = 0$, $H^1(X; G) = \sum_{e_\alpha \in A} G$

Lemma: If $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact, then

$$Hom(A, G) \xleftarrow{f^*} Hom(B, G) \xleftarrow{g^*} Hom(C, G) \leftarrow 0 \text{ is exact.}$$

Proof:

Claim: g onto implies g^* is 1:1.

Suppose $g^*(\psi) = \psi \circ g = 0$. Since g is onto, $\psi(x) = 0$ for all $x \in C$. Thus $\psi = 0$ and g^* is 1:1.

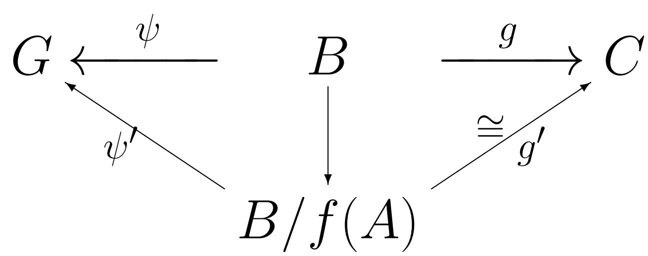
Thus we have exactness at $Hom(C, G)$.

Claim: $im(f) = ker(g)$ implies $im(g^*) = ker(f^*)$.

$im(f) \subset ker(g)$ implies $g \circ f = 0$
implies $f^* \circ g^* = (g \circ f)^* = 0^* = 0$ and thus $im(g^*) \subset ker(f^*)$.

Suppose $\psi \in ker(f^*)$, $\psi : B \to G$. Then $f^*(\psi) = \psi \circ f = 0$. Thus $\psi(f(A)) = 0$ and ψ induces homomorphism $\psi' : B/f(A) \to G$

g induces an isomorphism $g' : B/ker(g) = B/f(A) \to C$.



$$g^*(\psi' \circ (g')^{-1}) = \psi' \circ (g')^{-1} \circ g = \psi$$

Lemma: If $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is split exact, then

$0 \to Hom(A,G) \xleftarrow{f^*} Hom(B,G) \xleftarrow{g^*} Hom(C,G) \leftarrow 0$ is split exact.

Proof: $\exists \pi : B \to A$ such that $\pi \circ f = id_A$.

Thus $(\pi \circ f)^* = f^* \circ \pi^* =$ identity on $Hom(A,G)$.

Thus f^* is surjective and the dual sequence splits.

Note also that $Hom(\bigoplus A_\alpha, G) = \prod Hom(A_\alpha, G)$,

and thus $Hom(A \bigoplus C, G) = Hom(A,G) \bigoplus Hom(C,G)$

Example: The dual of the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \to 0$

$$0 \leftarrow Hom(\mathbb{Z},G) \xleftarrow{t^*} Hom(\mathbb{Z},G) \xleftarrow{\pi^*} Hom(\mathbb{Z}_2,G) \leftarrow 0$$

$\pi^*(\psi) = \psi \circ \pi : \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \xrightarrow{\psi} G$, defined by $(\psi \circ \pi)(1) = \psi(1)$.

$t^*(\psi) = \psi \circ t : \mathbb{Z} \xrightarrow{t} \mathbb{Z} \xrightarrow{\psi} G,$

defined by $(\psi \circ t)(1) = \psi(2) = \psi(1) + \psi(1) = 2\psi(1)$.

Defn: A **free resolution** of an abelian group H is an exact sequence of abelian groups,

$$\ldots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where each F_i is free.

Recall an exact sequence is a chain complex, and the dual of a chain complex is a chain complex.

Thus the dualization of this free resolution is a chain complex:

$$\ldots \xleftarrow{f_2^*} Hom(F_1, G) \xleftarrow{f_1^*} Hom(F_0, G) \xleftarrow{f_0^*} Hom(H, G) \leftarrow 0$$

Let $H^n(F; G) = Ker(f_{n+1}^*)/im(f_n^*)$

Lemma 3.1: a.) Given two free resolutions F and F' of H and H', respectively, every homomorphism $\alpha : H \to H'$ can be extended to a chain map from F to F':

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & F_2 & \longrightarrow & F_1 & \xrightarrow{\phi} & F_0 & \xrightarrow{\psi} & H & \longrightarrow 0 \\
& \exists\alpha_2 \downarrow & & \exists\alpha_1 \downarrow & & \exists\alpha_0 \downarrow & & \downarrow \alpha & \\
\ldots \longrightarrow & F_2' & \longrightarrow & F_1' & \xrightarrow{\phi} & F_0' & \xrightarrow{\psi} & H' & \longrightarrow 0
\end{array}$$

Furthermore, any two such chain maps extending α are chain homotopic.

b.) For any two free resolutions F and F' of H, $\exists$ canonical isomorphism $H^n(F; G) = H^n(F', G)$ for all n.

Example: A short exact sequence of abelian groups,

$$0 \to F_1 \to F_0 \to H \to 0$$

where F_i are free is called a **free resolution of** H.

Example: $0 \to B_p(X) \hookrightarrow Z_p(X) \to H_p(X) \to 0$

Example:

Let $F_0 =$ the free abelian group generated by the generators of H.

Let $F_1 =$ kernel of projection map $F_0 \to H$.

Dual of the exact seq $0 \to F_1 \to F_0 \to H \to 0$ is the chain complex:

$$0 \xleftarrow{f_2^*} Hom(F_1, G) \xleftarrow{f_1^*} Hom(F_0, G) \xleftarrow{f_0^*} Hom(H, G) \leftarrow 0$$

Recall $F_1 \to F_0 \to H \to 0$ exact implies its dual is also exact:

$$Hom(F_1, G) \xleftarrow{f_1^*} Hom(F_0, G) \xleftarrow{f_0^*} Hom(H, G) \leftarrow 0$$

Note $H^n(F; G) = Ker(f_{n+1}^*)/im(f_n^*) = 0$ for $n > 1$.

And $H^0(F; G) = Ker(f_1^*)/im(f_0^*) = 0$.

But $H^1(F; G) = Ker(f_2^*)/im(f_1^*) =\ ?$.

Definition: $Ext(H, G) = H^1(F; G)$ (the extension of G by H).

For computational purposes, the following properties are useful.

(a) $\text{Ext}(H \oplus H', G) \cong \text{Ext}(H, G) \oplus \text{Ext}(H', G)$ since the direct sum of free resolutions is the free resolution of the direct sum.

(b) $\text{Ext}(H, G) = 0$ if H is free since $0 \to H \to H \to 0$ is a free resolution of H.

(c) $\text{Ext}(\mathbb{Z}/n, G) \cong G/nG$ by dualizing the free resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$.

to produce the exact sequence:

$$0 \leftarrow Ext(\mathbb{Z}_n, G) \leftarrow Hom(\mathbb{Z}, G) \xleftarrow{n} Hom(\mathbb{Z}, G) \leftarrow Hom(\mathbb{Z}_n, G) \leftarrow 0$$

Theorem 1. *If a chain complex $C_\bullet$ of free abelian groups has homology groups $H_\bullet(C)$, then the cohomology groups $H^\bullet(C; G)$ of the cochain complex* $\text{Hom}(C_\bullet, G)$ *are determined by the split exact sequences*

$$0 \longrightarrow \text{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C; G) \xrightarrow{h} \text{Hom}(H_n(C), G) \longrightarrow 0.$$

Corollary 1. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_n$, then*

$$H^n(C; Z) \cong (H_n/T_n) \oplus T_{n-1}.$$

Corollary 2. *If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G.*