

The following diagram does NOT commute:

$$\begin{array}{ccccccc}
 \cdots & \rightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \rightarrow & \cdots \\
 & & \nearrow f_{n+1} & \parallel g_{n+1} & \searrow K_n & \parallel f_n & \parallel g_n & \searrow K_{n-1} & \parallel f_{n-1} & \parallel g_{n-1} & \nearrow \\
 \cdots & \rightarrow & D_{n+1} & \xrightarrow{\partial'_{n+1}} & D_n & \xrightarrow{\partial'_n} & D_{n-1} & \rightarrow & \cdots
 \end{array}$$

The chain maps $f_n, g_n : C_n \rightarrow D_n$ are chain homotopic if $\exists K_n : C_n \rightarrow D_n$ such that

$$\partial'_{n+1}K_n + K_{n-1}\partial_n = f_n - g_n$$

Claim: $f_* = g_* : H_\bullet^C \rightarrow H_\bullet^D$

Proof: If $\alpha \in Z_n^C$, then $\partial\alpha = 0$.

$$\partial'_{n+1}K_n(\alpha) + K_{n-1}\partial_n(\alpha) = f_n(\alpha) - g_n(\alpha)$$

$$f_n(\alpha) = \partial'_{n+1}K_n(\alpha) + g_n(\alpha)$$

$$[f_n(\alpha)] = [\partial'_{n+1}K_n(\alpha) + g_n(\alpha)] = [g_n(\alpha)]$$

Thus $f_*(\alpha) = g_*(\alpha)$