NOTE: all maps are assumed to be continuous unless otherwise stated.

$r : X \twoheadrightarrow A$ is a *retraction of X onto A*

if $r|_A =$ identity map on A.

A is a *retract* of X if $\exists$ a retraction of X onto A.

Ex: The constant map $c : X \to x_0$, where x_0 is a point of X is a retraction.

A is a *deformation retract* of X if the identity map $i : X \to X$ is homotopic to a map $R : X \to X$ where R is the extension (of the codomain) of a retraction $r : X \to A$ and where each point of A remains fixed during the homotopy.

In other words, $\exists$ homotopy $H : X \times I \to X$ such that $H_0 =$ identity map on X, $H_1(X) \subset A$, and $H_t|_A =$ identity map on A $\forall\, t$.

Ex: X is a deformation retract of $X \times I$.

Ex: S^n is a deformation retract of $R^{n+1} - \{\mathbf{0}\}$.

Lemma 55.1: If A is a retract of X, then

$$i_* : \pi_1(A) \to \pi_1(X) \text{ is injective}$$

where i_* is induced by the inclusion map.

Lemma 58.3: If A is a deformation retract of X, then

$$i_* : \pi_1(A) \to \pi_1(X) \text{ is an isomorphism}$$

where i_* is induced by the inclusion map.

Thm 70.1: Seifert-van Kampen Theorem.

Suppose U, V and $U \cap V$ are open and path-connected. Let $i : U \cap V \to U$ and $j : U \cap V \to V$ be inclusion maps. If

$$\pi_1(U) = < a_1, ..., a_i \mid s_1, ..., s_l >,$$

$$\pi_1(V) = < b_1, ..., b_j \mid t_1, ..., t_m >,$$

$$\pi_1(U \cap V) = < c_1, ..., c_k \mid r_1, ..., r_n >,$$

then $\pi_1(U \cup V) =$

$$< a_1, ..., a_i, b_1, ..., b_j \mid s_1, ..., s_l, t_1, ..., t_m, i(c_1) = j(c_1), ..., i(c_n) = j(c_n) >$$