Let $\mathcal{U} = \{U_\alpha\}$ such that $X \subset \cup U_\alpha^o$.

Then $C_n^{\mathcal{U}}(X) = \{\sum r_i\sigma_i \mid \sigma_i \subset U_\alpha \text{ for some } \alpha\}$ is a subgroup of $C_n(X)$.

$\partial(C_n^{\mathcal{U}}(X)) \subset C_{n-1}^{\mathcal{U}}(X)$ and $\partial^2 = 0$. Thus $\exists H_n^{\mathcal{U}}(X)$

Prop 2.2.1: The inclusion map $i : C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain homotopy equivalence.

I.e., $\exists\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ suth that $i\rho$ and ρi are chain homotopic to the identity.

Hence i induces an isomorphism $H_n^{\mathcal{U}}(X) \cong H_n(X)$.

(1) Barycentric subdivision of of (ideal) simplices.

Simplex $[v_0, ..., v_n] = \{\sum t_i v_i \mid \sum t_i = 1, t_i \geq 0\}$

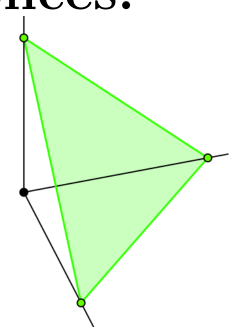


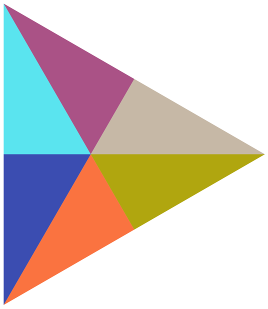
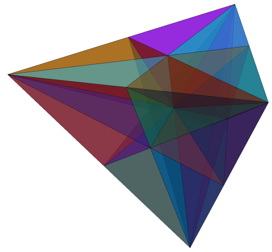
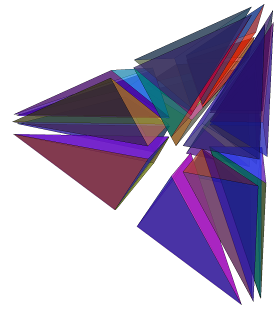
Figure 1: http://www.wikiwand.com/en/Simplex

The barycenter = center of gravity = $b = \sum_{i=0}^{n} \frac{1}{n+1} v_i$

Barycentric subdivision: decompose $[v_0, ..., v_n]$ into the $n-$simplices $[b, w_0, ..., w_{n-1}]$, inductively.

Divide each edge $[v_1, v_2]$ in half, forming 2 new edges $[b, v_1]$, $[b, v_2]$.

Note: $diam[b, v_i] = ||v_i - b|| = \frac{1}{2}||v_2 - v_1|| = \frac{1}{2}(diam[v_1, v_2])$

http://drorbn.net/AcademicPensieve/2010-06/

Claim:
If b is a barycenter of $[v_0, ..., v_{k-1}]$, then $||b - v_i|| \le \left(\frac{k-1}{k}\right) ||v_j - v_k||$.

Thus diam $[b, w_0, ..., w_{k-1}] \le \left(\frac{k-1}{k}\right) diam[v_0, ..., v_n]$

Note: Claim is true for $k = 2$. Suppose claim is true for $k = n - 1$.

Suppose all the faces of $[v_0, ..., v_n]$ have been subdivided. For all $n-1$-simplices $[w_0, ..., w_{n-1}]$ in this subdivision, form the $n-$simplices $[b, w_0, ..., w_{n-1}]$, where b is the barycenter of $[v_0, ..., v_n]$

By induction $||w_i - w_j|| \le \left(\frac{n-1}{n}\right) ||v_l - v_k||$.

Let b_i be the barycenter of $[v_0, ..., \widehat{v_i}, ..., v_n]$

$$b = \sum_{j=0}^{n} \frac{1}{n+1} v_j = \left(\frac{1}{n+1}\right) v_i + \sum_{j \ne i} \left(\frac{1}{n+1}\right) v_j = \left(\frac{1}{n+1}\right) v_i + \left(\frac{n}{n+1}\right) \sum_{j \ne i} \left(\frac{1}{n}\right) v_j$$

$$= \left(\frac{1}{n+1}\right) v_i + \left(\frac{n}{n+1}\right) b_i$$

Thus $||b - v_i|| = \left(\frac{n}{n+1}\right) ||b_i - v_i|| \le \left(\frac{n}{n+1}\right) ||v_j - v_i||$

Thus $diam[b, w_0, ..., w_{n-1}] \le \left(\frac{n}{n+1}\right) diam[v_0, ..., v_n]$

Thus repeated barycentric subdivision leads to simplices whose diameter is arbitrarily small.

2. Barycentric subdivision of Linear Chains

For Y convex, define $LC_n(Y) = \{\lambda : \Delta^n \to Y \mid \lambda \text{ is linear }\}$

$\partial(LC_n(Y)) \subset LC_{n-1}(Y)$.

For convenience, define $LC_{-1}(Y) = \mathbb{Z} =< [\emptyset] >$ where $\partial[v] = [\emptyset]$

If $b \in Y$, define homomorphism $b : LC_n(Y) \to LC_{n+1}(Y)$, $b([w_0, ..., w_n]) = [b, w_0, ..., w_n]$, the cone operator.

$\partial b([w_0, ..., w_n]) = \partial[b, w_0, ..., w_n] = [w_0, ..., w_n] - b\partial[w_0, ..., w_n]$.

Thus if $\alpha = \sum_{i=1}^{n} r_i\lambda_i$, then $(\partial \circ b)(\alpha) = \alpha - (b \circ \partial)(\alpha), \forall \alpha \in LC_n(Y)$.

$(\partial \circ b)(\alpha) + (b \circ \partial)(\alpha) = \alpha$

That is $\partial \circ b + b \circ \partial = id - 0$, where id = the identity homomorphism and 0 = the constant zero homomorphism on $LC_n(Y)$.

Thus b is a chain homotopy between the identity map and the zero homomorphism on the augmented chain complex $LC(Y)$.

Define subdivision homomorphism $S : LC_n(Y) \to LC_n(Y)$ by induction on n.

Let $\lambda : \Delta^n \to Y$ be a generator of $LC_n(Y)$.

Let $b_\lambda = \lambda(b)$ where b is the barycenter of Δ^n.

Define $S([\emptyset]) = [\emptyset]$ and $S(\lambda) = b_\lambda(S(\partial(\lambda)))$

Ex: If $\lambda = [v]$, then $b_\lambda = v$ and

$$S([v]) = b_\lambda(S(\partial([v]))) = v(S([\emptyset])) = v([\emptyset]) = [v].$$

Thus S is the identity on $LC_{-1}(Y)$ and $LC_0(Y)$.

Ex: If $\lambda = [v, w]$, $S([v, w]) = b_\lambda(S(\partial([v, w])))$

$$= b_\lambda(S([w]) - S([v])) = b_\lambda([w] - [v]) = [b_\lambda, w] - [b_\lambda, v].$$

Ex: If $\lambda = [u, v, w]$, $S(u, [v, w]) = b_\lambda(S(\partial([u, v, w])))$

$= b_\lambda(S([v, w]) - S([u, w]) + S([u, v]))$

$= b_\lambda([b_{v,w}, w] - [b_{v,w}, v] - ([b_{u,w}, w] - [b_{u,w}, u]) + [b_{u,v}, v] - [b_{u,v}, u])$

$= [b_\lambda, b_{v,w}, w] - [b_\lambda, b_{v,w}, v] - [b_\lambda, b_{u,w}, w] + [b_\lambda, b_{u,w}, u] + [b_\lambda, b_{u,v}, v] - [b_\lambda, b_{u,v}, u]$

If λ is an embedding, $S(\lambda)$ is the alternating sum of the simplices in the barycentric subdivision of λ.

Claim: S is a chain homotopy between $LC_n(Y)$ and itself.

That is $\partial S = S\partial$.

Proof by induction on n:

True for $n = -1, 0$ since $S = id$.

$\partial(S(\lambda)) = \partial(b_\lambda(S(\partial(\lambda)))) = (1 - b_\lambda\partial)(S(\partial(\lambda))))$

$= S(\partial(\lambda)) - b_\lambda(\partial(S(\partial(\lambda)))) = S(\partial(\lambda)) - b_\lambda(S(\partial(\partial(\lambda))))$

$= S(\partial(\lambda)) - b_\lambda(S(0) = S(\partial(\lambda))$

Define a chain homotopy between S and id,

$$T : LC_n(Y) \to LC_{n+1}(Y) \text{ inductively:}$$

$T = 0$ for $n = -1$, and $T(\lambda) = b_\lambda(\lambda - T\partial\lambda)$.

Thus $T([v]) = v([v] - T\partial[v]) = v([v] - T[\emptyset]) = v([v]) = [v, v]$.

$T([v, w]) = b_\lambda([v, w] - T\partial[v, w]) = b_\lambda([v, w] - T([w] - [v]))$

$\quad = b_\lambda([v, w] - [w, w] + [v, v]) = [b_\lambda, v, w] - [b_\lambda, w, w] + [b_\lambda, v, v]$

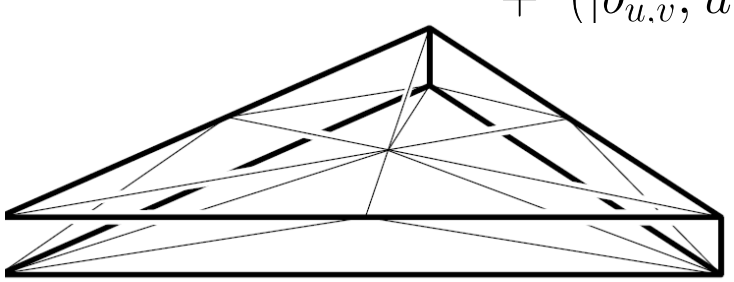
$T([u, v, w]) = b_\lambda([u, v, w] - T\partial[u, v, w])$

$$= b_\lambda([u, v, w] - T([v, w] - [u, w] + [u, v])$$

$$= b_\lambda([u, v, w] - ([b_{v,w}, v, w] - [b_{v,w}, w, w] + [b_{v,w}, v, v]$$

$$- \ ([b_{u,w}, u, w] - [b_{u,w}, w, w] + [b_{u,w}, u, u])$$

$$+ \ ([b_{u.v}, u, v] - [b_{u,v}, v, v] - [b_{u,v}, u, u])$$



from: Hatcher

$\partial(T(\lambda)) = \partial b_\lambda(\lambda - T\partial\lambda)$

$= \lambda - T\partial\lambda - b_\lambda\partial(\lambda - T\partial\lambda)$ since $\partial b_\lambda - id = b_\lambda\partial$

$= \lambda - T\partial\lambda - b_\lambda[\partial\lambda - \partial T(\partial\lambda)]$ since ∂ is a homomorphism.

$= \lambda - T\partial\lambda - b_\lambda[S(\partial\lambda) - T\partial(\partial\lambda)]$ by $id - \partial T = S - T\partial$ for dim(n-1).

$= \lambda - T\partial\lambda - b_\lambda[S(\partial\lambda)]$ since $\partial^2 = 0$

$= \lambda - T\partial\lambda - S(\lambda)$ since $S(\lambda) = b_\lambda(S(\partial(\lambda)))$

Thus $\partial T(\lambda) = \lambda - T\partial(\lambda) - S(\lambda)$. I.e., $\partial T + T\partial = id - S$.

In other words, T is a chain homotopy between id and S.

3. Barycentric subdivision of general chains:

Currently S is only defined on convex subsets Y.

For example: $S : C_n(\Delta^n) \to C_n(\Delta^n)$.

For example if $n = 1$, $\Delta^n = [v, w]$ with barycenter b_λ, then

$$S(id_{[v,w]}) = id_{[b_\lambda,w]} - id_{[b_l,v]}$$

We can extend S to $C_n(X)$ as follows:

$$S : C_n(X) \to C_n(X) \text{ by } S(\sigma) = \sigma_\# S(\Delta^n).$$

For example, if $\sigma : [v, w] \to X \ \in \ C_n(X)$ with barycenter b_λ,

$$S(\sigma) = \sigma_\# S(\Delta^n) = \sigma \circ (id_{[b_\lambda,w]} - id_{[b_\lambda,v]}) = \sigma_{[b_\lambda,w]} - \sigma_{[b_\lambda,v]}.$$

Note $\partial S = S\partial$:

$\partial(S\sigma) = (\partial\sigma_{\#})S\Delta^n = \sigma_{\#}(\partial S)\Delta^n = \sigma_{\#}S(\partial\Delta^n)$

$= \sigma_{\#}S(\sum_i(-1)^i\Delta_i^n)$ by defn of ∂ where Δ_i^n is the ith face of Δ^n

$= \sum_i(-1)^i\sigma_{\#}S(\Delta_i^n)$, since $\sigma_{\#}$ and S are homomorphisms.

$= \sum_i(-1)^iS(\sigma|_{\Delta_i^n})$ by defn of S.

$= S(\sum_i(-1)^i(\sigma|_{\Delta_i^n}))$ since S is a homomorphism.

$= S(\partial\sigma)$ by defn of $\partial\sigma$

Similarly, extend $T : C_n(X) \to C_{n+1}(X)$ by $T(\sigma) = \sigma_{\#}T(\Delta^n)$.

For example, if $\sigma : [v, w] \to X \in C_n(X)$ with barycenter b_λ,

$T(\sigma) = \sigma_{\#}T(\Delta^n) = \sigma \circ (b_\lambda([v, w] - T\partial[v, w]))$

$= \sigma \circ (b_\lambda([v, w] - T([w] - [v])))$

$= \sigma \circ (b_\lambda([v, w] - [w, w] + [v, v]))$

$= \sigma \circ ([b_\lambda, v, w] - [b_\lambda, w, w] + [b_\lambda, v, v])$

$= \sigma|_{[b_\lambda,v,w]} - \sigma|_{[b_\lambda,w,w]} + \sigma|_{[b_l,v,v]}$.

T is a chain homotopgy between S and id.

$$\partial T\sigma = \partial \sigma_{\#} T(\Delta^n) = \sigma_{\#} \partial T(\Delta^n) = \sigma_{\#}(\Delta^n - S\Delta^n - T\partial \Delta^n)$$

$$= \sigma - S\sigma - T(\partial \sigma)$$

Hence $\partial T + T\partial = id - S$.

4. Iterated Barycentric subdivision

$D_m : C_n(X) \to C_{n+1}(X)$ defined by

$$D_m = \sum_{i=0}^{m-1} TS^i \qquad \text{is a chain homotopy between } id \text{ and } S^m\text{:}$$

$$\partial D_m + D_m \partial = \partial(\sum_{i=0}^{m-1} TS^i) + (\sum_{i=0}^{m-1} TS^i)\partial = \sum_{i=0}^{m-1} (\partial TS^i + TS^i \partial)$$

$$= \sum_{i=0}^{m-1} (\partial TS^i + T\partial S^i) = \sum_{i=0}^{m-1} (\partial T + T\partial)S^i = \sum_{i=0}^{m-1} (id - S)S^i$$

$$= id - S^m.$$

Let $\mathcal{U} = \{U_\alpha\}$ such that $X \subset \cup U_\alpha^o$.

For each singular simplex $\sigma : \Delta^n \to X$, choose the smallest m_σ such that the diameter of the simplices of $S^{m_\sigma}(\Delta^n)$ is less than the Lebesgue number of the cover of Δ^n by $\{\sigma^{-1}(U_\alpha^o)\}$.

Define $D : C_n(X) \to C_{n+1}(X)$ by $D(\sigma) = D_{m_s}(\sigma)$

Define $\rho : C_n(X) \to C_n(X)$ by $\rho = id - \partial D - D\partial$.

ρ is a chain map:

$$\partial\rho(\sigma) = \partial\sigma - \partial\partial D\sigma - \partial D\partial\sigma = \partial\sigma - \partial D\partial\sigma.$$

$$\rho\partial(\sigma) = \partial\sigma - \partial D\partial\sigma - D\partial\partial\sigma = \partial\sigma - \partial D\partial\sigma.$$

Thus D is a chain homotopy between id and ρ.

Claim: $\rho(C_n(X)) \subset C_n^{\mathcal{U}}(X)$

$\rho(\sigma) = \sigma - \partial D\sigma - D\partial\sigma = \sigma - \partial D_{m_\sigma}\sigma - D\partial\sigma$

$= S^{m_\sigma}(\sigma) - D_{m_\sigma}\partial(\sigma) - D\partial\sigma$ since $id - \partial D_{m_\sigma} = S^{m_\sigma} - D_{m_\sigma}\partial$

$= S^{m_\sigma}(\sigma) - D_{m_\sigma}(\sum(-1)^i\sigma_i) - D(\sum(-1)^i\sigma_i)$
where σ_i is the ith face of σ

$= S^{m_\sigma}(\sigma) - D_{m_\sigma}(\sum(-1)^i\sigma_i) - D_{m_{\sigma_i}}(\sum(-1)^i\sigma_i)$

Since $\sigma_i \subset \sigma$, $m_{\sigma_i} \leq m_\sigma$. Thus each term is in $C_n^{\mathcal{U}}(X)$

Define $\rho' : C_n(X) \to C_n^{\mathcal{U}}(X)$ by $\rho' = \rho$. Then $\rho = i \circ \rho'$

Thus D is a chain homotopy between id and $i \circ \rho'$.

Note if $\sigma \in C_n^{\mathcal{U}}(X)$, then

$$D(\sigma) = (id - S^{m_\sigma})(\sigma) = (id - id)(\sigma) = 0.$$

Thus $\rho' = id - \partial D - D\partial = id$ and $\rho' \circ i$ is the identity on $C_n^{\mathcal{U}}(X)$

Thus ρ' is the chain homotopy inverse of i.