Degree

Let $f : S^n \to S^n$ for $n > 0$.

Then $f_* : H_n(S^n) = \mathbb{Z} \to \mathbb{Z} = H_n(S^n)$.

f_* is a homomorphism and thus $f_*(\alpha) = d\alpha$.

Defn: The *degree of* f is d.

a.) $deg\ id = 1$

b.) f not onto implies $deg\ f = 0$

Suppose $x_0 \in S^n - f(S^n)$. Then $S^n \to S^n - \{x_0\} \hookrightarrow S^n$ implies $f_* = 0$ since $H_n(S^n - \{x_0\} = 0$

c.) If f is homotopic to g, then $f* = g*$ and thus $deg\ f = deg\ g$.

Hopf Thm (cor 4.25): If $deg\ f = deg\ g$, then f is homotopic to g.

d.) $(f \circ g)_* = f_* \circ g_*$, and thus $deg\ (f \circ g) = (deg\ f)(deg\ g)$

e.) Let $S^n = \{x \in R^{n+1} \mid ||x|| = 1\}$. $deg\ r_i = -1$ where

$r_i(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_{n+1}) = (x_1, ..., x_{i-1}, -x_i, x_{i+1}, ..., x_{n+1})$.

$S^n = \Delta_1^n \bigcup_{\partial} \Delta_2^n, \quad H_n(S^n) =< \Delta_1^n - \Delta_2^n >$

and $f(\Delta_1^n - \Delta_2^n) = -\Delta_1^n + \Delta_2^n$

f.) The antipodal map $-id : S^n \to S^n$, $\;-id(x) = -x$

has degree $(-1)^{n+1}$ since $r_1 \circ r_2 \circ ... \circ r_{n+1} = -id$.

g.) If $f : S^n \to S^n$ has no fixed points, then $deg\ f = (-1)^{n+1}$ since f is homotopic to $-id$ via the homotopy

$$F(x,t) = \frac{(1-t)f(x) - tx}{||(1-t)f(x) - tx||}$$

If $(1-t)f(x) - tx = 0$, then $f(x) = (\frac{t}{1-t})x$

$x, f(x) \in S^n$ implies $\frac{t}{1-t} = 1, -1$.

But if $f(x) = -x$, then $(1-t)f(x) - tx = (1-t)(-x) - tx = -x$.

Thus $(1-t)f(x) - tx = 0$ iff f has a fixed point and thus F is well-defined if f has no fixed points.

h.) If $Sf : S^{n+1} \to S^{n+1}$, $S([x,t]) = S([f(x),t])$ denotes

the suspension map of $f : S^n \to S^n$, then $\deg Sf = \deg f$.

The cone of of $S^n = CS^n = (S^n \times I)/(S^n \times 1)$

with base $S^n = S^n \times 0 \subset CS^n$.

$S^{n+1} =$ the suspension $SS^n = CS^n/S^n$

$$H_{n+1}(CS^n) \to H_{n+1}(CS^n, S^n) \xrightarrow{\partial_*} H_n(S^n) \to H_n(CS^n)$$

i.) $f : S^1 \to S^1$, $f(z) = z^k$ has degree k.

Thus $S^{n-1}f : S^n \to S^n$ has degree k

Suppose $f : S^n \to S^n$ and $\exists y$ such that $f^{-1}(y) = \{x_1, ..., x_m\}$.

Choose $U_l,\ V$ open such that $x_l \in U_l$, $y \in V$, $f(U_l) \subset V$.

Then $f(U_l - x_l) \subset V - y$ and the following diagram commutes:

$$\begin{tikzcd}
 & H_n(U_l, U_l - x_l) \arrow[r, "f_*"] \arrow[d, "i_{U_l*}"'] \arrow[dl] & H_n(V, V - y) \arrow[d, "\cong"'] \\
H_n(S^n, S^n - x_l) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "i_*"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[u, "j"] \arrow[r, "f_*"] \arrow[ul] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

$f_* : H_n(U_l, U_l - x_l) = \mathbb{Z} \to \mathbb{Z} = H_n(V, V - y)$, $f_*(\alpha) = d_l \alpha$.

Defn: The *local degree* of f at $x_l = deg\ f|_{x_l} = d_l$.

Prop: $deg\ f = \sum_{l=1}^{m} deg\ f|_{x_l}$

$$H_n(S^n,\ S^n - f^{-1}(y)) \cong H_n(\sqcup U_l,\ \sqcup U_l - f^{-1}(y))$$
$$= \oplus H_n(U_l,\ U_l - x_l) = \oplus \mathbb{Z}.$$

$(i_* \circ j)(1) = 1$. Thus $j(1) = (1, 1, ..., 1) = \sum i_{U_l*}(1)$

$f_* \circ j(1) = (1, 1, ..., 1) = \sum f_* \circ i_{U_l*}(1) = \sum d_l$

Note: If $f : U_l \to V$ is a homeomorphism, then $deg\ f|_{x_l} = \pm 1$

Theorem 2.28: A continuous nonvanishing vector field on S^n exists if and only if n is odd.

Proof: ($\Rightarrow$) Suppose $\exists$ a continuous nonvanishing vector field, v, on S^n

Normalize the vector field so that $|v(x)| = 1$ for all x.

Then $v(x) \in S^n$ and $v(x)$ is perpendicular to x.

Thus $(cos(\pi t))x + (sin(\pi t))v(x) \in S^n$.

Then $F(x,t) = (cos(\pi t))x + (sin(\pi t))v(x)$ is a homotopy between the identity map on S^n and the antipodal map.

Thus $1 = (-1)^{n+1}$ and n is odd.

($\Leftarrow$) Let $v(x_1, x_2, ..., x_{2l-1}, x_{2l}) = (-x_2, x_1, ..., -x_{2l}, x_{2l-1})$

Proposition 2.29: If n is even, then $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n.

Suppose G acts on S^n. Then $g \in G$ defines a homeomorphism $g : S^n \to S^n$. Since g is a homeomorphism, $deg\ g = \pm 1$.

$d : G \to \{\pm 1\}$, $d(g) = deg\ g$ is a homomorphism by property d.

If the action is free, then if $g \neq e$, $d(g) = (-1)^{n+1}$ by property g.

Thus if n is even, $g \neq e$ implies $d(g) = -1$, Thus $ker(d) = e$ and d is an isomorphism. Thus $G \cong \{\pm 1\} \cong \mathbb{Z}_2$